

Thinking the Unthinkable: The Story of Complex Numbers (with a Moral)

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THE USUAL DEFINITION of complex numbers, either as ordered pairs (a, b) of real numbers or as “numbers” of the form $a + bi$, does not give any indication of their long and tortuous evolution, which lasted about three hundred years. I want to describe this evolution very briefly because I think some lessons can be learned from this story, just as from many other such stories concerning the evolution of a concept, result, or theory. These lessons have to do with the impact of the history of mathematics on our understanding of mathematics and on our effectiveness in teaching it. But more about the moral of this story later.

Birth

This story begins in 1545. What came earlier can be summarized by the following quotation from Bhaskara, a twelfth-century Hindu mathematician (Dantzig 1967):

The square of a positive number, also that of a negative number, is positive; and the square root of a positive number is two-fold, positive and negative; there is no square root of a negative number, for a negative number is not a square.

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Jerome Cardan (1501–1576)

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In 1545 Jerome Cardan, an Italian mathematician, physician, gambler, and philosopher, published a book entitled *Ars Magna* (The great art), in which he described an algebraic method for solving cubic and quartic equations. This book was a great event in mathematics. It was the first major achievement in algebra since the time, 3000 years earlier, when the Babylonians showed how to solve quadratic equations. Cardan, too, dealt with quadratics in his book. One of the problems he proposed is the following (Struik 1969):

If some one says to you, divide 10 into two parts, one of which multiplied into the other shall produce . . . 40, it is evident that this case or question is impossible. Nevertheless, we shall solve it in this fashion.

Cardan then applied his algorithm (essentially the method of completing the square) to $x + y = 10$ and $xy = 40$ to get the two numbers $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Moreover, "putting aside the mental tortures involved" (Burton 1985), Cardan formally multiplied $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ and obtained 40. He did not pursue the matter but concluded that the result was "as subtle as it is useless" (NCTM 1969). Although eventually rejected, this event was nevertheless historic, since it was the first time ever that the square root of a negative number was explicitly written down. And, as Dantzig (1985) has observed, "the mere writing down of the impossible gave it a symbolic existence."

In the solution of the cubic equation, square roots of negative numbers had to be reckoned with. Cardan's solution for the cubic $x^3 = ax + b$ was given as

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\left(\frac{b}{2}\right) - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}},$$

the so-called Cardan formula. When applied to the historic example $x^3 = 15x + 4$, the formula yields

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Although Cardan claimed that his general formula for the solution of the cubic was inapplicable in this case (because of the appearance of $\sqrt{-121}$), square roots of negative numbers could no longer be so lightly dismissed. Whereas for the quadratic (e.g., $x^2 + 1 = 0$) one could say that no solution exists, for the cubic $x^3 = 15x + 4$ a real solution, namely $x = 4$, does exist; in fact, the two other solutions, $-2 \pm \sqrt{3}$, are also real. It now remained to reconcile the formal and "meaningless" solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

of $x^3 = 15x + 4$, found by using Cardan's formula, with the solution $x = 4$, found by inspection. The task was undertaken by the hydraulic engineer Rafael Bombelli about thirty years after the publication of Cardan's work.

Bombelli had the "wild thought" that since the radicands $2 + \sqrt{-121}$ and $2 - \sqrt{-121}$ differ only in sign, the same might be true of their cube roots. Thus, he let

$$\sqrt[3]{2 + \sqrt{-121}} = a + \sqrt{-b}$$

and

$$\sqrt[3]{2 - \sqrt{-121}} = a - \sqrt{-b}$$

and proceeded to solve for a and b by manipulating these expressions according to the established rules for real variables. He deduced that $a = 2$ and $b = 1$ and thereby showed that, indeed,

$$\begin{aligned} & \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\ &= (2 + \sqrt{-1}) + (2 - \sqrt{-1}) = 4 \end{aligned}$$

(Burton 1985). Bombelli had thus given meaning to the "meaningless." This event signaled the birth of complex numbers. In his own words (*Leapfrogs* 1980):

It was a wild thought, in the judgement of many; and I too was for a long time of the same opinion. The whole matter seemed to rest on sophistry rather than on truth. Yet I sought so long, until I actually proved this to be the case.

Of course, breakthroughs are achieved in this way—by thinking the unthinkable and daring to present it in public.

The equation $x^3 = 15x + 4$ represents the so-called irreducible case of the cubic, in which all three solutions are real yet they are expressed (by Cardan's formula) by means of complex numbers. To resolve the apparent paradox of cubic equations exemplified by this type of equation, Bombelli developed a calculus of operations with complex numbers. His rules, in our symbolism, are $(\pm 1)i =$

$\pm i$, $(+i)(+i) = -1$, $(-i)(+i) = +1$, $(\pm 1)(-i) = \mp i$, $(+i)(-i) = +1$, and $(-i)(-i) = -1$. He also considered examples involving addition and multiplication of complex numbers, such as $8i + (-5i) = +3i$ and

$$(\sqrt[3]{4 + \sqrt{2}i})(\sqrt[3]{3 + \sqrt{8}i}) = \sqrt[3]{8 + 11\sqrt{2}i}.$$

Bombelli thus laid the foundation stone of the theory of complex numbers.

Many textbooks, even at the university level, suggest that complex numbers arose in connection with the solution of quadratic equations, especially the equation $x^2 + 1 = 0$. As indicated previously, the cubic rather than the quadratic equation forced the introduction of complex numbers.

Growth

Bombelli's work was only the beginning of the saga of complex numbers. Although his book *L'Algebra* was widely read, complex numbers were shrouded in mystery, little understood, and often entirely ignored. Witness Simon Stevin's remark in 1585 about them (Crossley 1980):

There is enough legitimate matter, even infinitely much, to exercise oneself without occupying oneself and wasting time on uncertainties.

Similar doubts concerning the meaning and legitimacy of complex numbers persisted for two and a half centuries. Nevertheless, during that same period complex numbers were extensively used and a considerable amount of theoretical work was done. We illustrate this work with a number of examples.

As early as 1620, Albert Girard suggested that an equation of degree n may have n roots. Such statements of the fundamental theorem of algebra were, however, vague and unclear. For example, René Descartes, who coined the unfortunate word "imaginary" for the new numbers, stated that although one can imagine that every equation has as many roots as is indicated by its degree, no (real)

numbers correspond to some of these imagined roots.

The following quotation, from a letter in 1673 from Christian Huygens to Gottfried von Leibniz in response to the latter's letter that contained the identity

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} = \sqrt{6},$$

was typical of the period (Crossley 1980):

The remark which you make concerning . . . imaginary quantities which, however, when added together yield a real quantity, is surprising and entirely novel. One would never have believed that

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}}$$

make $\sqrt{6}$ and there is something hidden therein which is incomprehensible to me.



Gottfried Wilhelm von Leibniz (1646–1716)

From *A Portfolio of Eminent Mathematicians*, ed. David Eugene Smith (Chicago: Open Court, 1896)

Leibniz, who spent considerable time and effort on the question of the meaning of complex numbers and the possibility of deriving reliable results by applying the ordinary laws of algebra to them, thought of them as "a fine and wonderful refuge of the divine spirit—almost an amphibian between being and non-being" (Leapfrogs 1980).

Complex numbers were widely used in the eighteenth century. Leibniz and John Bernoulli

used imaginary numbers as an aid to integration. For example,

$$\begin{aligned}\int \frac{1}{x^2 + a^2} dx &= \int \frac{1}{(x + ai)(x - ai)} dx \\ &= -\frac{1}{2ai} \int \left(\frac{1}{x + ai} - \frac{1}{x - ai} \right) dx \\ &= -\frac{1}{2ai}\end{aligned}$$

$[\log(x + ai) - \log(x - ai)]$. This use, in turn, raised questions about the meaning of the logarithm of complex as well as negative numbers. A heated controversy ensued between Leibniz and Bernoulli. Leibniz claimed, for example, that $\log i = 0$, arguing that $\log(-1)^2 = \log 1^2$, and hence $2 \log(-1) = 2 \log 1 = 0$; thus $\log(-1) = 0$, and hence $0 = \log(-1) = \log i^2 = 2 \log i$, from which it follows that $\log i = 0$. Bernoulli opted for $\log i = (\pi i)/2$; this equation follows from Euler's identity $e^{\pi i} = -1$, which implies that $\log(-1) = \pi i$ and hence that $\log i = \frac{1}{2} \log(-1) = (\pi i)/2$, although this argument is not the one that Bernoulli used. The controversy was subsequently resolved by Leonhard Euler (*Leapfrogs* 1978).

Complex numbers were used by Johann Lambert for map projection, by Jean D'Alembert in hydrodynamics, and by Euler, D'Alembert, and Joseph-Louis Lagrange in incorrect proofs of the fundamental theorem of algebra. (Euler, by the way, was the first to designate $\sqrt{-1}$ by i .)

Euler, who made fundamental use of complex numbers in linking the exponential and trigonometric functions by the formula $e^{ix} = \cos x + i \sin x$, expressed himself about them in the following way (Kline 1972):

Because all conceivable numbers are either greater than zero, less than zero or equal to zero, then it is clear that the square root of negative numbers cannot be included among the possible numbers.... And this circumstance leads us to the concept of such numbers, which by their nature are impossible and ordinarily are called imaginary or fancied numbers, because they exist only in the imagination.

Even the great Carl Friedrich Gauss, who in his doctoral thesis of 1797 gave the first essentially



Leonhard Euler (1707–1783)

From *A Portfolio of Eminent Mathematicians*, ed. David Eugene Smith (Chicago: Open Court, 1908)

correct proof of the fundamental theorem of algebra, claimed as late as 1825 that “the true metaphysics of $\sqrt{-1}$ is elusive” (Kline 1972).

It should be pointed out that the desire for a logically satisfactory explanation of complex numbers became manifest in the latter part of the eighteenth century, on philosophical, if not on utilitarian, grounds. With the advent of the Age of Reason in the eighteenth century, when mathematics was held up as a model to be followed, not only in the natural sciences but in philosophy as well as political and social thought, the inadequacy of a rational explanation of complex numbers was disturbing.

The problem of the logical justification of the laws of operation with negative and complex numbers also became a pressing pedagogical issue at, among other places, Cambridge University at the turn of the nineteenth century. Since mathematics was viewed by the educational institutions as a paradigm of rational thought, the glaring inadequacies in the logical justification of the operations with negative and complex numbers became untenable. Such questions as, “Why does $2 \times i + i = 2$?” and “Is $\sqrt{ab} = \sqrt{a} \sqrt{b}$ true for negative a and b ?” received no satisfactory answers. In fact, Euler, in his text of the 1760s on algebra, claimed $\sqrt{-1} \sqrt{-4} = \sqrt{4} = +2$ as a possible result. Robert Woodhouse opined in 1802 that since imaginary numbers lead to right conclusions, they must have



Karl Friedrich Gauss (1777–1855)

From *A Portfolio of Eminent Mathematicians*, ed. David Eugene Smith (Chicago: Open Court, 1908)

a logic. Around 1830 George Peacock and others at Cambridge set for themselves the task of determining that logic by codifying the laws of operation with numbers. Although their endeavor did not satisfactorily resolve the problem of the complex numbers, it was perhaps the earliest instance of “axiomatics” in algebra.

By 1831 Gauss had overcome his scruples concerning complex numbers and, in connection with a work on number theory, published his results on the geometric representation of complex numbers as points in the plane. Similar representations by the Norwegian surveyor Caspar Wessel in 1797 and by the Swiss clerk Jean-Robert Argand in 1806 went largely unnoticed. The geometric representation, given Gauss’s stamp of approval, dispelled much of the mystery surrounding complex numbers. In the next two decades further development took place. In 1833 William Rowan Hamilton gave an essentially rigorous algebraic definition of complex numbers as pairs of real numbers. (To Hamilton the complex number (a, b) consisted of a pair of “moments of time,” since he had earlier defined real numbers, under Immanuel

Kant’s influence, as “moments of time.”) In 1847 Augustin-Louis Cauchy gave a completely rigorous and abstract definition of complex numbers in terms of congruence classes of real polynomials modulo $x^2 + 1$. In this, Cauchy modeled himself on Gauss’s definition of congruences for integers (Kline 1972).

Maturity

By the latter part of the nineteenth century all vestiges of mystery and distrust of complex numbers could be said to have disappeared, although a lack of confidence in them persisted among some textbook writers well into the twentieth century. These authors would often supplement proofs using imaginary numbers with proofs that did not involve them. Complex numbers could now be viewed in the following ways:

1. Points or vectors in the plane
2. Ordered pairs of real numbers
3. Operators (i.e., rotations of vectors in the plane)
4. Numbers of the form $a + bi$, with a and b real numbers
5. Polynomials with real coefficients modulo $x^2 + 1$
6. Matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$
 with a and b real numbers
7. An algebraically closed, complete field
(This is an early twentieth-century view.)

Although the preceding various ways of viewing the complex numbers might seem confusing rather than enlightening, it is of course commonplace in mathematics to gain a better understanding of a given concept, result, or theory by viewing it in as many contexts and from as many points of view as possible.

The foregoing descriptions of complex numbers are not the end of the story. Various developments in mathematics in the nineteenth century enable us to gain a deeper insight into the role of



William Rowan Hamilton (1805–1865)

complex numbers in mathematics and in other areas. Thus, complex numbers offer just the right setting for dealing with many problems in mathematics in such diverse areas as algebra, analysis, geometry, and number theory. They have a symmetry and completeness that is often lacking in such mathematical systems as the integers and real numbers. Some of the masters who made fundamental contributions to these areas say it best: The following three quotations are by Gauss in 1801, Riemann in 1851, and Hadamard in the 1890s, respectively:

Analysis . . . would lose immensely in beauty and balance and would be forced to add very hampering restrictions to truths which would hold generally otherwise, if . . . imaginary quantities were to be neglected. (Birkhoff 1973)

The origin and immediate purpose of the introduction of complex magnitudes into mathematics lie in the theory of simple laws of dependence between variable magnitudes expressed by means of operations on magnitudes. If we enlarge the scope of applications of these laws by assigning to the variables they involve complex values, then there appears an otherwise hidden harmony and regularity. (Ebbinghaus 1983)

The shortest path between two truths in the real domain passes through the complex domain. (Kline 1972)

The descriptions of such developments are rather technical. Only the barest of illustrations can be given:

(1) In algebra, the solution of polynomial equations motivated the introduction of complex numbers: Every equation with complex coefficients has a complex root—the so-called fundamental theorem of algebra. Beyond their use in the solution of algebraic polynomial equations, the complex numbers offer an example of an algebraically closed field, relative to which many problems in linear algebra and other areas of abstract algebra have their “natural” solution.

(2) In analysis, the nineteenth century saw the development of a powerful and beautiful branch of mathematics, namely complex function theory. We have already seen how the use of complex numbers gave us deeper insight into the logarithmic, exponential, and trigonometric functions. Moreover, we can evaluate real integrals by means of complex function theory. One indication of the efficacy of the theory is that a function in the complex domain is infinitely differentiable if once differentiable. Such a result is, of course, false in the case of functions of a real variable (e.g., $f(x) = x^{4/3}$).

(3) The complex numbers lend symmetry and generality in the formulation and description of various branches of geometry, for example, Euclidean, inversive, and non-Euclidean. Thus, by the introduction of ideal points into the plane any two circles can now be said to intersect at two points. This idea aids in the formulation and proof of many results. As another example, Gauss used the complex numbers to show that the regular polygon of seventeen sides is constructible with straightedge and compass.

(4) In number theory, certain Diophantine equations can be solved neatly and conceptually by the use of complex numbers. For example, the equation $x^2 + 2 = y^3$, when expressed as $(x + \sqrt{2}i)(x - \sqrt{2}i) = y^3$, can readily be solved, in integers, using properties of the complex domain consisting of the set of elements of the form $a + b\sqrt{2}i$, with a and b integers.

(5) An elementary illustration of Hadamard's dictum that "the shortest path between two truths in the real domain passes through the complex domain" is supplied by the following proof that the product of sums of two squares of integers is again a sum of two squares of integers; that is,

$$(a^2 + b^2)(c^2 + d^2) = u^2 + v^2,$$

for some integers u and v . For,

$$\begin{aligned} (a^2 + b^2)(c^2 + d^2) &= (a + bi)(a - bi)(c + di)(c - di) \\ &= [(a + bi)(c + di)][(a - bi)(c - di)] \\ &= (u + vi)(u - vi) \\ &= u^2 + v^2. \end{aligned}$$

Try to prove this result without the use of complex numbers and without being given the u and v in terms of a, b, c , and d .

In addition to their fundamental uses in mathematics, some of which were previously indicated, complex numbers have become a fixture in science and technology. For example, they are used in quantum mechanics and in electric circuitry. The "impossible" has become not only possible but indispensable.

The Moral

Why the history of mathematics? Why bother with such "stories" as this one? Edwards (1974) puts it in a nutshell:

Although the study of the history of mathematics has an intrinsic appeal of its own, its chief *raison d'être* is surely the illumination of mathematics itself.

My colleague Abe Shenitzer expresses it as follows:

One can *invent* mathematics without knowing much of its history. One can *use* mathematics without knowing much, if any, of its history. But one cannot

have a mature appreciation of mathematics without a substantial knowledge of its history.

Such appreciation is essential for the teacher to possess. It can provide him or her with insight, motivation, and perspective—crucial ingredients in the making of a good teacher. Of course, whether this story has succeeded in achieving these objectives in relation to the complex numbers is for the reader to judge. However, beyond the immediate objective of lending insight, this story and others like it may furnish us with a slightly better understanding of the nature and evolution of the mathematical enterprise. It addresses such themes or issues as the following:

(1) *The meaning of number in mathematics.* Complex numbers do not fit readily into students' notions of what a number is. And, of course, the meaning of number has changed over the centuries. This story presents a somewhat better perspective on this issue. It also leads to the question of whether numbers beyond the complex numbers exist.

(2) *The relative roles of physical needs and intellectual curiosity as motivating factors in the development of mathematics.* In this connection it should be pointed out that the problem of the solution of the cubic, which motivated the introduction of complex numbers, was *not* a practical problem. Mathematicians already knew how to find approximate roots of cubic equations. The issue was to find a theoretical algebraic formula for the solution of the cubic—a question without any practical consequences. Yet how useful did the complex numbers turn out to be! This is a recurring theme in the evolution of mathematics.

(3) *The relative roles of intuition and logic in the evolution of mathematics.* Rigor, formalism, and the logical development of a concept or result usually come at the end of a process of mathematical evolution. For complex numbers, too, first came *use* (theoretical rather than practical), then *intuitive understanding*, and finally *abstract justification*.

(4) *The nature of proof in mathematics.* This question is related to the preceding item. But although (3) addresses the evolution of complex numbers in its broad features, this item deals with local questions of proof and rigor in establishing various results about complex numbers (cf., e.g., the derivation of the value of $\log i$ by von Leibniz and Bernoulli). One thing is certain—what was acceptable as a proof in the seventeenth and eighteenth centuries was no longer acceptable in the nineteenth and twentieth centuries. The concept of proof in mathematics has evolved over time, as it is still evolving, and not necessarily from the less to the more rigorous proof (cf. the recent proof, by means of the computer, of the four-color conjecture). Philip Davis goes a step further in outlining the evolution of mathematical ideas (Davis 1965):

It is paradoxical that while mathematics has the reputation of being the one subject that brooks no contradictions, in reality it has a long history of successful living with contradictions. This is best seen in the extensions of the notion of number that have been made over a period of 2500 years. From limited sets of integers, to infinite sets of integers, to fractions, negative numbers, irrational numbers, complex numbers, transfinite numbers, each extension, in its way, overcame a contradictory set of demands.

(5) *The relative roles of the individual and the environment in the creation of mathematics.* What was the role of Bombelli as an individual in the creation of complex numbers? Cardan surely had the opportunity to take the great and courageous step of “thinking the unthinkable.” Was the time perhaps not ripe for Cardan, but ripe for Bombelli about thirty years later? Is it the case, as John Bolyai stated, that “mathematical discoveries, like springtime violets in the woods, have their season which no human can hasten or retard” (Kline 1972)? This conclusion certainly seems to be borne out by many instances of independent and simultaneous discoveries in mathematics, such as the geometric representation of complex numbers by



George Pólya (1887–1985)

Courtesy of Birkhäuser Boston

Wessell, Argand, and Gauss. The complex numbers are an interesting case study of such questions, to which, of course, we have no definitive answers.

(6) *The genetic principle in mathematics education.* What are the sources of a given concept or theorem? Where did it come from? Why would anyone have bothered with it? These are fascinating questions, and the teacher should at least be aware of the answers to such questions. When and how he or she uses them in the classroom is another matter. On this matter George Pólya (1962) says the following:

Having understood how the human race has acquired the knowledge of certain facts or concepts, we are in a better position to judge how the human child should acquire such knowledge.

Can we not at least have a better appreciation of students' difficulties with the concept of complex numbers, having witnessed mathematicians of the first rank make mistakes, “prove” erroneous theorems, and often come to the right conclusions for insufficient or invalid reasons?

Some Suggestions for the Teacher

Let me conclude with some comments on, and suggestions for, the use of the history of mathematics in the teaching of mathematics, in particular with reference to complex numbers. Many of the points are implicit in the preceding story.

(1) I first want to reiterate what I view as the major contribution of this story for the teacher. Pólya (1962) puts it very well:

To teach effectively a teacher must develop a feeling for his subject; he cannot make his students sense its vitality if he does not sense it himself. He cannot share his enthusiasm when he has no enthusiasm to share. How he makes his point may be as important as the point he makes; he must personally feel it to be important.

The objective of my story, then, is to give the teacher some feeling for complex numbers, to imbue him or her with some enthusiasm for complex numbers.

When it comes to suggestions for classroom use, it cannot be overemphasized that these are only suggestions. The teacher, of course, can better judge when and how, at what level, and in what context to introduce and relate historical material to the discussion at hand. The introduction of historical material can, however, convey to the student the following important lessons, which are usually not imparted through the standard curriculum.

(2) Mathematics is far from a static, lifeless discipline. It is dynamic, constantly evolving, full of failures as well as achievements.

(3) Observation, analogy, induction, and intuition are the initial and often the more natural ways of acquiring mathematical knowledge. Rigor and proof usually come at the end of the process.

(4) Mathematicians usually create their subject without thought of practical applications. The latter, if any, come later, sometimes centuries later. This point relates to "immediate relevance" and to "instant gratification," which students often seek from any given topic presented in class.

(5) We must, of course, supply the student with "internal relevance" when introducing a given concept or result. This point brings us to the important and difficult issue of motivation. To some students the applications of a theorem are appealing; to others, the appeal is in the inner logical structure of the theorem. A third factor, useful but often neglected, is the source of the theorem: How did it arise? What motivated mathematicians to introduce it? With complex numbers, their origin in the solution of the cubic, rather than the quadratic, should be stressed. Cardan's attempted division of ten into two parts whose product is forty reinforces this point. How much further one continues with the historical account is a decision better made by the teacher in the classroom, bearing in mind the lessons that should be conveyed through this or similar historical material.

(6) Historical projects deriving from this story about complex numbers can be given to able students as topics for research and presentation to, say, a mathematics club. Possible topics are the following:

(a) The logarithms of negative and complex numbers.

(b) What is a number? That is, discuss the evolution of various number systems and the evolution of our conception of what a number is.

(c) Hypercomplex numbers (e.g., the quaternions). Their discovery is another fascinating story.

(d) Gauss's congruences of integers and Cauchy's congruences of polynomials. The latter lead to a new definition (description) of complex numbers.

(e) An axiomatic characterization of complex numbers (see (7) under the heading "Maturity"). In this connection we ought to discuss the notion of characterizing a mathematical system, and thus the concept of isomorphism. (Cf. the various equivalent descriptions of complex numbers discussed previously.)

(7) Many elementary and interesting illustrations of Hadamard's comment demonstrate that "the shortest path between two truths in the real

domain passes through the complex domain." We are referring to elementary results from various branches of mathematics, results whose statements do not contain complex numbers but whose "best" proofs often use complex numbers. One such example was given previously. Some others by Cell (1950), Jones (1954), and the NCTM (1969) can be found in the Bibliography.

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