Full Complexity Analysis of the Diameter-Constrained Reliability

Eduardo Canale[a], Héctor Cancela[a], Franco Robledo[a], Pablo Romero[a] and Pablo Sartor[a]

[a]*Instituto de Computación, Facultad de Ingeniería, Universidad de la República., Julio Herrera y Reissig 565, CP 11300, Montevideo, Uruguay. Tel. (+598) 2711 4244 - Fax (+598) 2711 0469., Email:* $\{$*canale,cancela,frobledo,promero,psartor*$\}$*@fing.edu.uy*

Abstract. Let $G = (V, E)$ be a simple graph with $|V| = n$ nodes and $|E| = m$ links, a subset $K \subseteq V$ of *terminals*, a vector $p = (p_1, \ldots, p_m) \in [0, 1]^m$ and a positive integer d, called *diameter*. We assume nodes are perfect but links fail stochastically and independently, with probabilities $q_i = 1 - p_i$. The *diameter-constrained reliability* (DCR for short), is the probability that the terminals of the resulting subgraph remain connected by paths composed by d links, or less. This number is denoted by $R^d_{K,G}(p)$.

The general DCR computation is inside the class of $\mathcal{NP}$-Hard problems, since is subsumes the complexity that a random graph is connected.

The contributions of this paper are two-fold. First, a full analysis of the computational complexity of DCR-subproblems is presented in terms of the number of terminal nodes $k = |K|$ and diameter d.

Second, we extend the class of graphs that accept efficient DCR computation. In this class we include graphs with bounded co-rank, graphs with bounded genus, planar graphs, and, in particular, Monma graphs, which are relevant in robust network design.

Keywords: Network Reliability, Computational Complexity, Diameter-Constrained Reliability, Monma Graphs.

1 Introduction

The definition of DCR has been introduced in 2001 by Héctor Cancela and Louis Petingi, inspired in delay-sensitive applications over the Internet infrastructure (Cancela and Petingi, 2001). Nevertheless, its applications over other fields of knowledge enriches the motivation of this problem in the research community (Colbourn, 1999).

We wish to communicate special nodes in a network, called *terminals*, by d hops or less, in a scenario where nodes are perfect but links fail stochastically and independently. The all-terminal case with $d = n - 1$ is precisely the probability that a random graph is connected, or *classical reliability problem* (CLR for short). Arnon Rosenthal proved that the CLR is inside the class of $\mathcal{NP}$-Hard problems (Rosenthal, 1977). As a corollary, the general DCR is $\mathcal{NP}$-Hard as well, hence intractable unless $\mathcal{P} = \mathcal{NP}$.

The focus of this paper is the computational complexity of DCR subproblems in terms of the number of terminals k and diameter d, and the efficient computation of the DCR for distinguished graph topologies.

In Section 2, a formal definition of DCR is provided as a particular instance of a coherent stochastic binary system. The computational complexity of the DCR is discussed in terms of the diameter and number of terminals in Section 3. The contributions of this paper are two-fold. First, we close the complexity analysis of the DCR problem in terms of k and d. Indeed, we prove in Section 4 that the DCR is in the computational class of $\mathcal{NP}$-Hard problems in the all-terminal scenario ($k = n$) with a given diameter $d \geq 2$. The computational complexity for other possible pairs for k and d is already available from prior literature from this area.

Second, we extend the class of known graphs that permit an efficient (i.e. polynomial time) computation for the DCR in Section 5. A particular but relevant family of these graphs are Monma graphs, which plays a key role in the design of robust network design (Monma et al., 1990; Robledo, 2005; Canale et al., 2009). Concluding remarks and open problems are summarized in Section 6.

2 Terminology

We are given a system with m components. These components are either "up" or "down", and the binary state is captured by a binary word $x = (x_1, \ldots, x_m) \in \{0,1\}^m$. Additionally, we have a structure function $\phi : \{0,1\}^m \to \{0,1\}$ such that $\phi(x) = 1$ if the system works under state x, and $\phi(x) = 0$ otherwise. When the components work independently and stochastically with certain probabilities of operation $p = (p_1, \ldots, p_m)$, the pair (ϕ, p) defines a *stochastic binary system*, or SBS for short, following the terminology from Ball (1986). An SBS is *coherent* whenever $x \leq y$ implies that $\phi(x) \leq \phi(y)$, where the partial order set $(\leq, \{0,1\}^m)$ is bit-wise (i.e. $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in \{1, \ldots, m\}$). If $\{X_i\}_{i=1,\ldots,m}$ is a set of independent binary random variables with $P(X_i = 1) = p_i$ and $X = (X_1, \ldots, X_m)$, then $r = E(\phi(X)) = P(\phi(X) = 1)$ is the *reliability* of the SBS.

Now, consider a simple graph $G = (V, E)$, a subset $K \subseteq V$ and a positive integer d. A subgraph $G_x = (V, E_x) \subseteq G$ is *d-K-connected* if $d_x(u, v) \leq d, \forall \{u, v\} \subseteq K$, where $d_x(u, v)$ is the distance between nodes u and v in the graph G_x. Let us choose an arbitrary order of the edge-set $E = \{e_1, \ldots, e_m\}$, $e_i \leq e_{i+1}$. For each subgraph $G_x = (V, E_x)$ with $E_x \subseteq E$, we identify a binary word $x \in \{0,1\}^m$, where $x_i = 1$ if and only if $e_i \in E_x$; this is clearly a bijection. Therefore, we define the structure $\phi : \{0,1\}^m \to \{0,1\}$ such that $\phi(x) = 1$ if G_x is d-K-connected, and $\phi(x) = 0$ otherwise. If we assume nodes are perfect but links fail stochastically and independently ruled by the vector $p = (p_1, \ldots, p_m)$, the pair (ϕ, p) is a coherent SBS. Its reliability, denoted by $R^d_{K,G}(p)$, is called *diameter constrained reliability*, or DCR for short. A particular case is $R^{n-1}_{K,G}(p)$, called *classical reliability*, or CLR for short.

In a coherent SBS, a *pathset* is a state x such that $\phi(x) = 1$. A *minpath* is a state x such that $\phi(x) = 1$ but $\phi(y) = 0$ for all $y < x$ (i.e. a minimal pathset). A *cutset* is a state x such that $\phi(x) = 0$, while a *mincut* is a state x such that $\phi(x) = 0$ but $\phi(y) = 1$ if $y > x$ (i.e. a minimal cutset). We will denote $\mathcal{O}^K_d(G)$ to the set of all d-K-connected subgraphs of a ground graph G.

We recall a bit of terminology coming from graph theory, which will be used throughout this treatment. A graph $G = (V, E)$ is *bipartite* if there exists a bipartition $V = V_1 \cup V_2$ such that $E \subseteq \{\{x, y\} : x \in V_1, y \in V_2\}$. A *vertex cover* in a graph $G = (V, E)$ is a subset $V' \subseteq V$ such that V' meets all links in E.

Given two graphs G_1 and G_2 with the same vertex set V, $f : V \rightarrow V$ is a K-*isomorphism* from G_1 to G_2 if it is an isomorphism that fixes the set K. In that case G_1 and G_2 are K-*isomorphic*. Given a simple graph $G = (V, E)$ and $e = \{x, y\} \in E$, an *elementary division* of e is a couple of edges $e_1 = \{x, z\}$ and $e_2 = \{z, y\}$ that replace e in G, where $z \notin V$. Two graphs G_1 and G_2 are *homeomorphic* if there exists a graph G such that G_1 and G_2 can be obtained from G by means of a sequence of elementary divisions. If $P = \{V_1, \ldots, V_c\}$ is a partition of V, the *quotient graph* is $G' = (P, E')$, where $\{V_i, V_j\} \in E'$ if and only if $i \neq j$ and there exists an edge from a vertex of V_i to a vertex of V_j in E. We say v_j is *reachable* from v_i either when $v_i = v_j$ or there is a path from v_i to v_j. In a simple graph G, reachability is an equivalence relation, and c, the number of classes in the quotient graph, is the number of *connected components*. Given a simple graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, its *rank* is $r(G) = n - c$, while its *co-rank* is $c(G) = m - n + c$. A *connected graph* verifies $c = 1$; then $r(G) = n - 1$ and $c(G) = m - n + 1$. In topological graph theory, the *genus of a graph* G is the least natural g such that G can be drawn without crossing itself in a surface with genus g. A *planar graph* verifies $g = 0$.

3 Computational Complexity

The class $\mathcal{NP}$ is the set of problems polynomially solvable by a non-deterministic Turing machine (Garey and Johnson, 1979). A problem is $\mathcal{NP}$-Hard if it is at least as hard as every problem in the set $\mathcal{NP}$ (formally, if every problem in $\mathcal{NP}$ has a polynomial reduction to the former). It is widely believed that $\mathcal{NP}$-Hard problems are intractable (i.e. there is no polynomial-time algorithm to solve them). An $\mathcal{NP}$-Hard problem is $\mathcal{NP}$-Complete if it is inside the class $\mathcal{NP}$. Stephen Cook proved that the joint satisfiability of an input set of clauses in disjunctive form is an $\mathcal{NP}$-Complete decision problem; in fact, the first known problem of this class (Cook, 1971). In this way, he provided a systematic procedure to prove that a certain problem is $\mathcal{NP}$-Complete. Specifically, it suffices to prove that the problem is inside the class $\mathcal{NP}$, and that it is at least as hard as an $\mathcal{NP}$-Complete problem. Richard Karp followed this hint, and presented the first 21 combinatorial problems inside this class (Karp, 1972). Leslie Valiant defines the class $\#\mathcal{P}$ of counting problems, such that testing whether an element should be counted or not can be accomplished in polynomial time (Valiant, 1979). A problem is $\#\mathcal{P}$-Complete if it is in the set $\#\mathcal{P}$ and it is at least as hard as any problem of that class.

Recognition and counting minimum cardinality mincuts/minpaths are at least as hard as computing the reliability of a coherent SBS (Ball, 1986). Arnon Rosenthal proved the CLR is $\mathcal{NP}$-Hard (Rosenthal, 1977), showing that the minimum cardinality mincut recognition is precisely Steiner-Tree problem, included in Richard Karp's list. The CLR for both two-terminal and all-terminal cases are still $\mathcal{NP}$-Hard, as Michael Ball and J. Scott Provan proved by reduction to counting minimum cardinality $s-t$ cuts (Provan and Ball, 1983). As a consequence, the general DCR is $\mathcal{NP}$-Hard as well. Later effort has been focused to particular cases of the DCR, in terms of the number of terminals $k=|K|$ and diameter d.

When $d=1$ all terminals must have a direct link, $R_{K,G}^{1}=\prod_{\{u,v\}\subseteq K}p(uv)$, where $p(uv)$ denotes the probability of operation of link $\{u,v\}\in E$, and $p(uv)=0$ if $\{u,v\}\notin E$. The problem is still simple when $k=d=2$. In fact, $R_{\{u,v\},G}^{2}=1-(1-p(uv))\prod_{w\in V-\{u,v\}}(1-p(uw)p(wv))$. Héctor Cancela and Louis Petingi rigorously proved that the DCR is $\mathcal{NP}$-Hard when $d\geq 3$ and $k\geq 2$ is a fixed input parameter (Cancela and Petingi, 2004), in strong contrast with the case $d=k=2$. Its proof is the main source of inspiration of this paper, and will be revisited in Section 4. The literature offers two proofs that the DCR has a polynomial-time algorithm when $d=2$ and k is a fixed input parameter (Sartor, 2013; Canale et al., 2013). Pablo Sartor et. al. present a recursive proof (Sartor, 2013), while Eduardo Canale et. al. present an explicit expression for $R_{K,G}^{2}$ that is computed in a polynomial time of elementary operations (Canale et al., 2014).

Fig. 1 summarizes the known results for the computational complexity of the DCR in terms of d and k.

4 Main theorem

The DCR is inside the class of $\mathcal{NP}$-Hard problems in the all-terminal case with diameter $d\geq 2$. We first prove the result when $d\geq 3$, and separately establish the case $d=2$.

The main source of inspiration for the first result is the article authored by Cancela and Petingi (2004), where they proved that the DCR is $\mathcal{NP}$-Hard when $d\geq 3$ and $k\geq 2$ is a fixed input parameter. There, the authors prove first that the result holds for $k=2$, and they further generalize the result for fixed $k\geq 2$. For our purpose it will suffice to revisit the first part.

d	k (fixed): 2	k (fixed): $3\dots$	$k = n$ or free
2	$O(n)$	$O(n)$	Unknown
$3, \dots, n-2$	$\mathcal{NP}$-Hard		Unknown
$n-1, \dots$	$\mathcal{NP}$-Hard		$\mathcal{NP}$-Hard

Figure 1: DCR Complexity in terms of the diameter d and number of terminals $k = |K|$

Before, we state a technical result:

Lemma 1. *Counting the number of vertex covers of a bipartite graph is #$\mathcal{P}$-Complete (Ball and Provan, 1983).*

Proposition 1. *The DCR is $\mathcal{NP}$-Hard when $k = 2$ and $d \geq 3$ (Cancela and Petingi, 2004).*

Proof. Let $d' = d - 3 \geq 0$ and $P = (V(P), E(P))$ a simple path with node set $V(P) = \{s, s_1, \dots, s_{d'}\}$ and edge set $E(P) = \{\{s, s_1\}, \{s_1, s_2\}, \dots, \{s_{d'-1}, s_{d'}\}\}$. For each bipartite graph $G = (V, E)$ with $V = A \cup B$ and $E \subseteq A \times B$ we build the following auxiliary network:

$$G' = (A \cup B \cup V(P) \cup \{t\}, E \cup E(P) \cup I\}, \quad (1)$$

where $I = \{\{s_{d'}, a\}, a \in A\} \cup \{\{b, t\}, b \in B\}$, and all links of G' are perfect but links in I, which fail independently with identical probabilities $p = 1/2$. Consider the terminal set $K = \{s, t\}$. The auxiliary graph G' is illustrated in Fig. 2. The reduction from the bipartite graph to the two-terminal instance is polynomial.

A vertex cover $A' \cup B' \subseteq A \cup B$ induces a cutset $I' = \{\{s_{d'}, a\}, a \in A'\} \cup \{\{b, t\}, b \in B'\}$ (i.e. if all links in I' fail, the nodes $\{s, t\}$ are not connected). Reciprocally, that cutset determines a vertex cover. Therefore, the number of cutsets $|\mathcal{C}|$ is precisely the number of vertex covers of

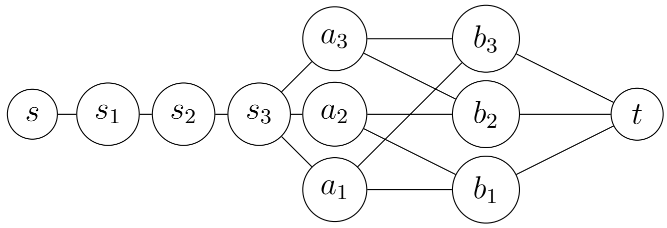

Figure 2: Example of auxiliary graph G'' with terminal set $\{s,t\}$ and $d=6$, for the bipartite instance C_6.

the bipartite graph $|\mathcal{B}|$. When $p = 1/2$, all cutsets are equally likely, and the source-terminal reliability evaluation at $p = 1/2$ is:

$$R^d_{\{s,t\},G'}(1/2) = 1 - \frac{|\mathcal{C}|}{2^{|A|+|B|}}$$

Finally, using the fact that $|\mathcal{B}| = |\mathcal{C}|$ and by substitution:

$$|\mathcal{B}| = 2^{|A|+|B|}(1 - R^d_{\{s,t\},G'}(1/2)).$$

Thus, the DCR for the two-terminal case is at least as hard as counting vertex covers of bipartite graphs. □

The result for $d \geq 3$ is a corollary of Proposition 1.

Theorem 1. *The DCR is $\mathcal{NP}$-Hard when $k = n$ and $d \geq 3$.*

Proof. Consider the auxiliary graph $G' = (V', E')$ from Fig. 2. Extend G' furthermore, and consider $G'' = (V'', E'')$, where $V'' = V'$ and $E'' = E' \cup \{\{a, a'\}, a \neq a', a, a' \in A\} \cup \{\{b, b'\}, b \neq b', b, b' \in B\}$. In words, just add links in order to connect all nodes from A, and all nodes from B. We keep the same probabilities of operation that in G', and the new links are perfect.

Consider now the all-terminal case $K = V''$ for G'', and given diameter $d \geq 3$. The key is to observe that the cutsets in the all-terminal scenario for G'' are precisely the $s-t$ cutsets in G', and they have the same probability.

Indeed, each pair of terminals from the set A are directly connected by perfect links; the same holds in B. The distance between s and $s_{d'}$ is $d' = d - 3 < d$, so these nodes (and all the intermediate ones) respect the diameter constraint. Finally, if there were an $s-t$ path (i.e.

a path from s to t), the diameter of G'' would be exactly d. Therefore, $R^d_{\{s,t\},G'} = R^d_{V'',G''}$, and again:

$$
\begin{aligned}
|\mathcal{B}| &= 2^{|A|+|B|}(1 - R^d_{\{s,t\},G'}(1/2)) \\
&= 2^{|A|+|B|}(1 - R^d_{V'',G''}(1/2)).
\end{aligned}
$$

Thus, the DCR for the all-terminal case is at least as hard as counting vertex covers of bipartite graphs. $\square$

Theorem 2. *The DCR is $\mathcal{NP}$-Hard when $k = n$ and $d = 2$.*

Proof. Given a graph $G = (V, E)$, consider $G' = (V \cup \{a, b\}, E \cup \{\{x, a\}, \{x, b\}, \forall x \in V\})$. By its definition, G' has diameter $d = 2$. All links are perfect, except the ones incident to a, with $p(ax) = 1/2$. Consider the DCR for G'. We will show that the number of minimum cardinality pathsets in G' is precisely the number of vertex covers in G'. Since counting minimum cardinality pathsets is at least as hard as computing the reliability of a coherent SBS (Ball, 1986), the result will follow.

A minimum cardinality pathset in G' contains all perfect links, and $\{a, x_1\}, \ldots, \{a, x_r\}$ for certain nodes $x_i \in V$. Since H is a minimum cardinality pathset, the graph $G_H = (V, H)$ has diameter 2, but the diameter is increased under any link deletion. Let $N_a = \{x : \{a, x\} \in H\}$ the set of neighbor vertices for the terminal node a. The key is to observe that *vertex a reaches every node in two steps if and only if N_a is a vertex cover.*

Indeed, suppose a reaches every node in two steps. Then, for any $x \in V \setminus N_a$ there exists a path xya, so $y \in N_a$ and thus N_a is a vertex cover. Conversely, if N_a covers V, let $x \in V$. Then, either $x \in N_a$ and x is adjacent with a, or $x \in V \setminus N_a$ and there exists $y \in N_a \cap N_x$, so xya is a path of two hops between x and a.

The minimality of N_a as a cover follows from the minimality of H as a pathset. $\square$

Theorems 1 and 2 jointly close the complexity analysis for the DCR problem. The whole picture of DCR complexity is provided in Fig. 3, which closes the complexity analysis for different independent pairs (k, d).

	k (fixed): 2	k (fixed): $3\dots$	$k=n$ or free
d: 2	$O(n)$	$O(n)$	$\mathcal{NP}$-Hard
d: $3, \vdots, n-2$	$\mathcal{NP}$-Hard		$\mathcal{NP}$-Hard
d: $n-1, \vdots$	$\mathcal{NP}$-Hard		$\mathcal{NP}$-Hard

Figure 3: DCR Complexity in terms of the diameter d and number of terminals $k = |K|$

5 Exact DCR Computation

So far, an efficient (polynomial-time) computation of the DCR is available only for special graphs, to know, paths, cycles, ladders, generalized ladders and spanish fans (Sartor, 2013). The reader can appreciate from Figure 3 that an efficient computation is also feasible for diameter $d = 2$ and a fixed number of input terminals k Canale et al. (2013). An explicit expression for $R_{K,G}^d(p)$ is provided by Canale et al. (2014).

We will include now a sketch of the proof. First, observe that $R_K(G, d)$ can be computed adding the probability of all pathsets

$$R_{K,G}^d = \sum_{G' \in \mathcal{O}_d^K(G)} p(G') \tag{2}$$

$$= \sum_{G' \in \mathcal{O}_d^K(G)} \prod_{e \in E(G')} p(e) \prod_{e \notin E(G')} (1 - p(e)). \tag{3}$$

If a link has non-terminal ends it can be removed without loss of generality when $d = 2$.

We will follow the terminology from Bollobás (2004), where $A^{(i)}$ represents the set of all subsets of A with exactly i elements. For instance, if $\{1, 2, 3\}^{(2)} = \{\{1, 2\}, \{1, 3\}, \{3, 2\}\}$. Two pathsets are *equivalent* if they are K-isomorphic. We will introduce an equivalence relation between the node set in order to rewrite Expression (2). Two vertices v and w are equivalent, and we write $v \equiv w$, if either they are the same or they are *twin vertices* in K^C, i.e., they do not belong to K and they are adjacent to exactly the same vertices. Now, let us consider the set of quotient graphs under $\equiv$.

Theorem 3. *Given a fixed k, it is possible to compute the diameter-constrained reliability in linear time on the order of the graph when the diameter is 2.*

Proof. We will show that *the number of non K-isomorphic quotient of minpaths is a function of k*. Therefore, this number is constant respect to the cardinal of n. Indeed, there are at most $2^{k(k-1)/2}$ ways to link vertices in K, and $2^k - 1$ possible non-twin vertices in K^C, thus we have at most $2^{\frac{k(k-1)}{2}} 2^{2^k - 1}$ possible non K-isomorphic quotient of minpaths. Notice that the set $\mathcal{O}_d^K(G)/\equiv$ of non K-isomorphic quotient of minpaths can be identified with the set of non K-isomorphic minpaths without twin vertices in K^C.

Therefore, in order to compute $R^2_{K,G}$ it is enough to run over the quotients Q in $\mathcal{O}_D^K(G)/\equiv$ and to find the probability that a subgraph G' has Q as a quotient, i.e.

$$
\begin{aligned}
R_K(G,2) &= \sum_{Q \in \mathcal{O}_d^K(G)/\equiv} p(\{G' : Q = (G'/\equiv)\}) \\
&= \sum_{Q \in \mathcal{O}_D^K(G)/\equiv} \quad \sum_{G' : Q = (G'/\equiv)} p(G'). \qquad (4)
\end{aligned}
$$

Without loss of generality, assume that G is a complete graph (just take $p(e) = 0$ if e is not part of the target graph). Let Q be a quotient graph, and $Q^{ext} := Q \setminus K$. Let us find the last sum in the last member of (4). For each subgraph G' such that $G'/\equiv = Q$, there is a function $f_{G'} : K^C \to Q^{ext} \cup \{a\}$ defined by

$$
f_{G'}(x) = \begin{cases} \textit{the class of } x \textit{ under} \equiv & \text{if } x \in V(G'), \\ a & \text{otherwise.} \end{cases}
$$

Conversely, given a function $f : K^C \to Q^{ext} \cup \{a\}$ such that every element of Q^{ext} has a preimage, there is a minpath G' with quotient Q such that $f = f_{G'}$.

We can generate all such functions considering the following polynomial form in the vari-

ables $P_{x,v}$ with $x \in K^C$ and $v \in Q^{ext}$:

$$F = \prod_{x \in K^C} (1 + \sum_{v \in Q^{ext}} P_{x,v}) \tag{5}$$

$$+ \sum_{i=1}^{|Q^{ext}|} (-1)^i \sum_{S \in (Q^{ext})^{(i)}} \prod_{x \in K^C} (1 + \sum_{v \in Q^{ext} \setminus S} P_{x,v}). \tag{6}$$

Each term of the polynomial F corresponds to a function $f_{G'}$, with the following interpretation: if $P_{x,v}$ is in a term, this means that the corresponding function maps x to v. Indeed, the first sum takes into account all the functions $f : K^C \to Q^{ext} \cup \{a\}$ without the restriction of covering Q^{ext}. The remaining summations apply Inclusion-Exclusion principle to take into account those functions that do not cover $i = 1, 2, \ldots, |Q^{ext}|$ vertices of Q^{ext}.

Now, we will compute the probability of each graph G', or equivalently, of the function $f_{G'}$. In order to do it, let us notice that the probability that a vertex $x \in K^C$ has image vertex $v \in Q^{ext}$ is the probability $p_{x,v}$ of vertex x to be adjacent to exactly the same vertices than v, which is

$$p_{x,v} = \prod_{y \in N_v} p(xy) \prod_{y \notin N_v} q(xy),$$

where N_v are the set of vertices adjacent with v. Besides, the probability of those edges between vertices in K is

$$p_Q = \prod_{e \in EQ \cap K^{(2)}} p(e) \prod_{e \in K^{(2)} \setminus EQ} q(e).$$

It is clear that p_Q depends only on Q, so is common to all graphs G' with quotient Q. Finally the probability of G' is:

$$p(G') = p_Q \prod_{x \in K^C} p_{x, f_{G'}(x)}.$$

All the computations can be done linearly on n.

But the last product is exactly the term in (5) corresponding to function $f_{G'}$ assigning the probabilities $p_{x,v}$ to the variables $P_{x,v}$. Thus, the second sum in (4) is p_Q times F evaluated in the probabilities $p_{x,v}$. Therefore,

$$R^2_{K,G} = \sum_{Q \in \mathcal{O}^K_D(G)/\equiv} p_Q \prod_{x \in K^C} (1 + \sum_{v \in Q^{ext}} p_{x,v})$$

$$+ \sum_{i=1}^{|Q^{ext}|} (-1)^i \sum_{s \in (Q^{ext})^{(i)}} \prod_{x \in K^C} (1 + \sum_{v \in Q^{ext} \setminus s} p_{x,v}),$$

This expression can be computed in linear time with respect to $n = |G|$. In fact, it is linear in the cardinal of K^C which is smaller than n, since the cardinal of Q^{ext} depends only on $|K|$. $\square$

We will further extend the previous list of graphs with efficient computation. The key is to bound the number of minpaths.

Theorem 4. *Given a connected graph $G = (V, E)$. If its co-rank $c(G) = |E| - |V| + 1$ has a constant bound, the diameter-constrained reliability can be computed in polynomial time.*

Proof. Given an arbitrary configuration $G' = (V, H) \subseteq G$, it is efficient to decide whether $G' \in \mathcal{O}_D^K(G)$ or not. Let us call $\mathcal{S}(G)$ to the set of *connected configurations of* G. Then

$$R_{K,G}^d = \sum_{G' \in \mathcal{S}(G)} 1_{\{G' \in \mathcal{O}_D^K(G)\}} \prod_{e \in E(G')} p(e) \prod_{e \notin E(G')} (1 - p(e)), \tag{7}$$

where $1_{\{x\}} = 1$ if x is true, and $1_{\{x\}} = 0$ otherwise. The cardinal of a configuration $G' \in \mathcal{S}(G)$ is at least $|V| - 1$, which corresponds to a spanning tree. Thus, the complement of G' in G has at most $|E| - |V| + 1$ links. The number of ways to choose these links give us the following bound

$$|\mathcal{S}(G)| \leq \binom{|E|}{|E| - |V| + 1}. \tag{8}$$

which is polynomial in $|V|$ if $c(G) = |E| - |V| + 1$, the co-rank of G, is a constant, since $|E| < |V|^2$. If the co-rank is bounded by h, then the diameter-constrained reliability can be found in $|E|^h |K||E| \leq |V|^{2h+3}$ elementary operations. $\square$

Corollary 1. *If the number of faces of a connected graph G of genus g has a constant bound, the diameter-constrained reliability can be computed in polynomial time.*

Proof. Follows from the fact that the number of faces f of a graph of genus g is $f = m - n + 2 - 2g = c(G) - (2g - 1)$. Then, the co-rank $c(G) = f + 2g - 1$ has a constant bound. $\square$

The relevance of the following corollary comes from the fact that most telecommunication networks are planar.

Corollary 2. *If the number of faces of a planar graph G has a constant bound, the diameter-constrained reliability can be computed in polynomial time.*

Proof. A planar graph has genus 0. □

The property is unaffected by elementary subdivisions of a graph:

Corollary 3. *If a family of graphs $\mathcal{F}$ has all its elements homeomorphic to a fixed graph, its diameter-constrained reliability can be computed in polynomial time with respect to its order.*

Proof. Homeomorphic graphs have the same co-rank. □

Now we focus on a distinguished family of graphs coming from robust network design. Specifically, Clyde Monma et. al. studied the minimum cost two-connected network design problem, for the metric case spanning all nodes in the set V Monma et al. (1990). There, the authors prove that there exists a solution $G' = (V, H) \subseteq G$ such that every vertex in G' has degree 2 or 3, and the deletion of one or two links from G' leaves one bridge in one of the resultant connected components. Moreover, those graphs are either a Hamiltonian cycle in G or contain a *Monma graph* as an induced subgraph. The term Monma graph was introduced in later works with this family of graphs Canale et al. (2009). Figure 4 sketches a general Monma graph.

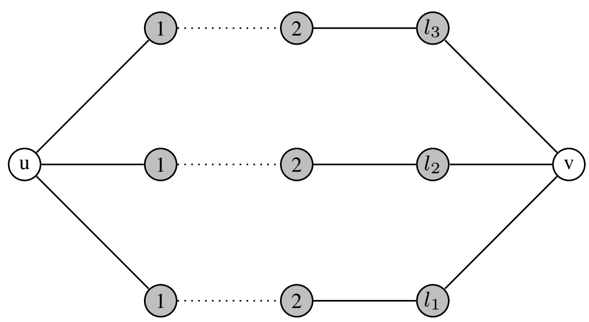

Figure 4: Monma's graph structure.

The following corollary provides the dimension of reliability in the study of Monma graphs:

Corollary 4. *The diameter-constrained reliability of Monma graphs can be computed in polynomial time respect to its order.*

Proof. Monma graphs are those homeomorphic to the general graph consisting in two vertices and three edges joining them. □

Finally, we further generalize Theorem 4.

Definition 1. *Let $G = (V, E)$ a simple graph, $K \subseteq V$ and d a positive integer. The graph G is d-K-r weak if $G - U$ is d-K disconnected, for every set $U \subseteq E$ with $|U| \geq r$.*

In words, "r-weakness" states the network fails (i.e. is not d-K connected) whenever we remove an arbitrary set of r links (or more).

Theorem 5. *Let $G = (V, E)$ a d-K-r weak graph, for some r independent of n. Then, the DCR can be found in polynomial time with $n = |V|$.*

Proof. Analogous to Theorem 4. Given an arbitrary configuration $G' = (V, H) \subseteq G$, we can decide in polynomial time whether G is d-K-connected or not. Let us denote $\mathcal{O}^r$ to the set of all configurations (V, H), with $|H| \geq m - r$, where $m = |E|$. By hypothesis we know that $\mathcal{O}_D^K(G) \subseteq \mathcal{O}^r$. Then, by Expression (7) we get that:

$$R_{K,G}^d = \sum_{G' \in \mathcal{O}^r} 1_{\{G' \in \mathcal{O}_D^K(G)\}} \prod_{e \in E(G')} p(e) \prod_{e \notin E(G')} (1 - p(e)), \tag{9}$$

It suffices to show that the number of terms in the sum is polynimial with respect to n. In fact, the cardinality $|\mathcal{O}^r|$ can be obtained with a sum-rule:

$$|\mathcal{O}^r| = \sum_{i=0}^{r-1} \binom{m}{m-i} \sim m^{r-1}, \tag{10}$$

where the symbol $\sim$ means that both sides are equivalent when m tends to infinity. Observe that $m < n^2$ holds for all connected graphs. Therefore, $|\mathcal{O}^r| \sim m^{r-1} \leq n^{2r-2}$, and the number of terms from Expression (9) is bounded by a polynomial in n, and $R_{K,G}^d$ can be found in a polynomial number of elementary operations in n. □

Remarks 1.

- *Theorem 5 is in fact a generalization of Theorem 4. Consider a simple graph $G = (V, E)$, with bounded co-rank $c(G)$, a terminal set $K \subseteq V$ and diameter d. If we delete an arbitrary link set $U \subseteq E$ of cardinality $c(G) + 1$, the resulting subgraph has less links than a tree. Then, $G - U$ us disconnected, and G is d-K-$(r + 1)$ weak.*

- *Let us show that Theorem 5 is a strict generalization. It suffices to find a d-K-r weak graph for some r independent of $n = |V|$ where its corank $c(G)$ is a function of n. Consider the complete bipartite graph $G = K_{(\frac{n}{2},\frac{n}{2})} = (V, E)$ in the all-terminal scenario $K = V$ with diameter $d = 2$. Its corank is $c(G) = \left(\frac{n}{2}\right)^2 - n + 1$ is a function of n. Thus, Theorem 4 cannot be applied. On the other hand, it is 2-V-1 weak. Indeed, consider an arbitrary link $e = \{x, y\} \in E$. Now, the shortest path between nodes x and y in $G - e$ has length three, hence it is not 2-V connected.*

- *A graph d-K-1 weak is d-K critical, in the sense that all spanning subgraphs are not d-K connected. The reliability of a d-K-1 weak graph $G = (V, E)$ is just the product of the reliabilities: $R^d_{K,G} = \prod_{e \in E} p_e$.*

6 Concluding Remarks

The computational complexity of a particular stochastic binary system has been discussed, called diameter constrained reliability (DCR). This measure joints quality with reliability, and it is the probability that all distinguished terminals $K \subseteq V$ in a network $G = (V, E)$ remain connected by d hops or less, where links $e \in E$ may fail with certain probabilities $q_e = 1 - p_e$.

When the number of terminals $k = |K|$ or diameter d are free, the DCR computation is $\mathcal{NP}$-Hard, since it subsumes the classical reliability problem. The cases $d = 1$ or $d = 2$ and k fixed belong to the set $\mathcal{P}$ of polynomially solvable problems; moreover, the DCR is found in a linear number of elementary operations in n for both cases.

In this paper we proved that the DCR is $\mathcal{NP}$-Hard for the remaining cases (i.e. where $k = n$ and $d \geq 3$). As a corollary, the result holds when $d \geq 2$ and k is a free parameter as well.

The family of graphs that accept a polynomial algorithm to find the DCR has been enriched in this work. To summarize, weak graphs, families with bounded co-rank and graphs with bounded number of faces are included in this family. In particular, we add trees, some critical bipartite graphs and Monma graphs as a corollary. In this way, this work connects reliability aspects of network design in a probabilistic context with robust network design. Indeed, the minimum-cost two-node connected subgraph of a metric graph G is either hamiltoninan or it presents a Monma graph as an induced subgraph.

As a future work, we would like to extend this family using other graph invariants, and apply these techniques to real-life scenarios, inspired in telecommunications. A challenging task is to extend the family of polynomial computations to special families of planar graphs (where the number of faces is a function of $n = |V|$), two and three-node connected networks, which are useful in robust network design. In particular, we are addressing the complexity of DCR computation in Halin graphs, where Theorems 4 and 5 cannot be applied.

References

Ball, M. O.: 1986, Computational complexity of network reliability analysis: An overview, *IEEE Transactions on Reliability* **35**(3), 230 –239.

Ball, M. O. and Provan, J. S.: 1983, The complexity of counting cuts and of computing the probability that a graph is connected, *SIAM J. Computing* **12**, 777–788.

Bollobás, B.: 2004, *Extremal Graph Theory*, Dover Books on Mathematics Series, Dover Publications.

Canale, E., Cancela, H., Robledo, F., Rubino, G. and Sartor, P.: 2013, On computing the 2-diameter-constrained K-reliability of networks, *International Transactions in Operational Research* **20**(1), 49–58.

Canale, E., Cancela, H., Robledo, F. and Sartor, P.: 2014, The complexity of computing the 2-k-reliability in networks, *Information Processing Letters* **114**(9), 457 – 461.

Canale, E., Monzón, P. and Robledo, F.: 2009, Global synchronization properties for different classes of underlying interconnection graphs for kuramoto coupled oscillators, *Future Generation Information Technology*, Vol. 5899 of *Lecture Notes in Computer Science*, Springer Berlin Heidelberg, pp. 104–111.

Cancela, H. and Petingi, L.: 2001, Diameter constrained network reliability: exact evaluation by factorization and bounds, *ICIL'2001 (International Conference on Industrial Logistics)*, pp. 359–366.

Cancela, H. and Petingi, L.: 2004, Reliability of communication networks with delay constraints: computational complexity and complete topologies, *International Journal of Mathematics and Mathematical Sciences* **2004**, 1551–1562.

Colbourn, C. J.: 1999, Reliability issues in telecommunications network planning, *Telecommu-*

nications network planning, chapter 9, Kluwer Academic Publishers, pp. 135–146.

Cook, S. A.: 1971, The complexity of theorem-proving procedures, *Proceedings of the third annual ACM symposium on Theory of computing*, STOC '71, ACM, New York, NY, USA, pp. 151–158.

Garey, M. R. and Johnson, D. S.: 1979, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman and Company, New York, NY, USA.

Karp, R. M.: 1972, Reducibility among combinatorial problems, *in* R. E. Miller and J. W. Thatcher (eds), *Complexity of Computer Computations*, Plenum Press, pp. 85–103.

Monma, C., Munson, B. and Pulleyblank, W.: 1990, Minimum-weight two-connected spanning networks, *Mathematical Programming* **46**(1-3), 153–171.

Provan, S. J. and Ball, M. O.: 1983, The Complexity of Counting Cuts and of Computing the Probability that a Graph is Connected, *SIAM Journal on Computing* **12**(4), 777–788.

Robledo, F.: 2005, *GRASP heuristics for Wide Area Network design*, PhD thesis, INRIA/IRISA, Université de Rennes I, Rennes, France.

Rosenthal, A.: 1977, Computing the reliability of complex networks, *SIAM Journal on Applied Mathematics* **32**(2), 384–393.

Sartor, P.: 2013, *Propriétés et méthodes de calcul de la fiabilité diamètre-bornée des réseaux*, PhD thesis, INRIA/IRISA, Université de Rennes I, Rennes, France.

Valiant, L.: 1979, The complexity of enumeration and reliability problems, *SIAM Journal on Computing* **8**(3), 410–421.