Review of Basic Coding Theory

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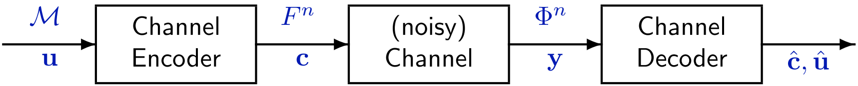
Channel Coding



Discrete probabilistic channel: (F, Φ, Prob)

- F: finite *input alphabet*, Φ: finite *output alphabet*
- Prob: conditional probability distribution

$$\text{Prob}\{\, \mathbf{y} \text{ received} \mid \mathbf{x} \text{ transmitted} \,\} \quad \mathbf{x} \in F^m, \quad \mathbf{y} \in \Phi^m, \quad m \geq 1$$

- $\mathbf{u}$: *message word* $\in \mathcal{M}$, set of M possible messages
- $\mathbf{c} \in F^n$: *codeword*, $\mathcal{E} : \mathbf{u} \xrightarrow{1-1} \mathbf{c}$ *encoding*
- $\mathcal{C} = \{\mathcal{E}(\mathbf{u}) \mid \mathbf{u} \in \mathcal{M}\}$ *code*
- $\mathbf{y} \in \Phi^n$: *received word*
- $\hat{\mathbf{c}}, \hat{\mathbf{u}}$: *decoded codeword, message word*, $\mathbf{y} \longrightarrow \hat{\mathbf{c}} \;\; (\longrightarrow \hat{\mathbf{u}})$ *decoding*

Code Parameters

$\mathcal{C} \;=\; \mathcal{E}(\mathcal{M}) \;\subseteq\; F^n, \quad |\mathcal{C}| = M$

- n: *code length*
- $k = \log_{|F|} M = \log_{|F|} |\mathcal{C}|$: *code dimension*
- $R = \dfrac{k}{n}$: *code rate* ≤ 1
- $r = n - k$: *code redundancy*
- We call $\mathcal{C}$ an (n, M) (*block*) *code* over F

The Hamming Metric

- *Hamming distance*

  For single-letters $x, y \in F$: $\mathsf{d}(x,y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$

  For vectors $\mathbf{x}, \mathbf{y} \in F^n$: $\mathsf{d}(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{n-1} \mathsf{d}(x_j, y_j)$

  number of locations where the vectors differ

- The Hamming distance defines a *metric*:
  - $\mathsf{d}(\mathbf{x},\mathbf{y}) \geq 0$, with equality if and only if $\mathbf{x} = \mathbf{y}$
  - Symmetry $\mathsf{d}(\mathbf{x},\mathbf{y}) = \mathsf{d}(\mathbf{y},\mathbf{x})$
  - Triangle inequality: $\mathsf{d}(\mathbf{x},\mathbf{y}) \leq \mathsf{d}(\mathbf{x},\mathbf{z}) + \mathsf{d}(\mathbf{z},\mathbf{y})$
- *Hamming weight* $\mathsf{wt}(\mathbf{e}) = \mathsf{d}(\mathbf{e}, \mathbf{0})$ *number of nonzero entries*
- When F is an abelian group, $\mathsf{d}(\mathbf{x},\mathbf{y}) = \mathsf{wt}(\mathbf{x} - \mathbf{y})$

Minimum Distance

- Let $\mathcal{C}$ be an (n, M) code over F, $M > 1$

$$d = \min_{\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C} \,:\, \mathbf{c}_1 \neq \mathbf{c}_2} \mathsf{d}(\mathbf{c}_1, \mathbf{c}_2)$$

  is called the *minimum distance* of $\mathcal{C}$
- We say that $\mathcal{C}$ is an (n, M, d) code.

Decoding

- $\mathcal{C} : (n, M, d)$ over F, used on channel $S = (F, \Phi, \text{Prob})$
- A *decoder* for $\mathcal{C}$ on S is a function

$$\mathcal{D} : \Phi^n \longrightarrow \mathcal{C}.$$

- *Decoding error probability* of $\mathcal{D}$ is

$$P_{\text{err}} = \max_{\mathbf{c} \in \mathcal{C}} P_{\text{err}}(\mathbf{c}) \ ,$$

where

$$P_{\text{err}}(\mathbf{c}) = \sum_{\mathbf{y} \,:\, \mathcal{D}(\mathbf{y}) \neq \mathbf{c}} \text{Prob}\{ \, \mathbf{y} \text{ received} \mid \mathbf{c} \text{ transmitted} \, \} \ .$$

goal: *find encoders (codes) and decoders that make P_{err} small*

Maximum Likelihood and Maximum a Posteriori Decoding

- $\mathcal{C} : (n, M, d)$, channel $S : (F, \Phi, \text{Prob})$.
- *Maximum likelihood decoder (MLD)*:

$$\mathcal{D}_{\text{MLD}}(\mathbf{y}) = \arg\max_{\mathbf{c}\in\mathcal{C}} \text{Prob}\{\ \mathbf{y} \text{ received} \mid \mathbf{c} \text{ transmitted}\ \},\ \forall \mathbf{y} \in \Phi^n$$

With a fixed tie resolution policy, $\mathcal{D}_{\text{MLD}}$ is well-defined for $\mathcal{C}$ and S.

- *Maximum a posteriori (MAP) decoder*:

$$\mathcal{D}_{\text{MAP}}(\mathbf{y}) = \arg\max_{\mathbf{c}\in\mathcal{C}} \text{Prob}\{\ \mathbf{c} \text{ transmitted} \mid \mathbf{y} \text{ received}\ \},\ \ \forall \mathbf{y} \in \Phi^n$$

But,

$$\begin{aligned} & \text{Prob}\{\,\mathbf{c} \text{ transmitted} \mid \mathbf{y} \text{ received}\,\} \\ & \quad = \quad \text{Prob}\{\,\mathbf{y} \text{ received} \mid \mathbf{c} \text{ transmitted}\,\} \cdot \frac{\text{Prob}\{\,\mathbf{c} \text{ transmitted}\,\}}{\text{Prob}\{\,\mathbf{y} \text{ received}\,\}} \end{aligned}$$

$\Longrightarrow$ MLD and MAP are the same when $\mathbf{c}$ is *uniformly distributed*

MLD on the BSC

- $\mathcal{C} : (n, M, d)$, channel $\mathsf{BSC}(p)$

$$\begin{aligned} &\mathsf{Prob}\{\,\mathbf{y}\ \mathsf{received}|\ \mathbf{c}\ \mathsf{transmitted}\,\} \\ &= \prod_{j=1}^{n} \mathsf{Prob}\{\,y_j\ \mathsf{received}\ |\ c_j\ \mathsf{transmitted}\,\} \\ &= p^{\mathsf{d}(\mathbf{y},\mathbf{c})}(1-p)^{n-\mathsf{d}(\mathbf{y},\mathbf{c})} = (1-p)^n \cdot \left(\frac{p}{1-p}\right)^{\mathsf{d}(\mathbf{y},\mathbf{c})}, \end{aligned}$$

where $\mathsf{d}(\mathbf{y}, \mathbf{c})$ is the Hamming distance. Since $p/(1-p) < 1$ for $p < 1/2$, for all $\mathbf{y} \in F_2^n$ we have

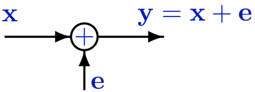
$$\mathcal{D}_{\mathrm{MLD}}(\mathbf{y}) = \arg\min_{\mathbf{c}\in\mathcal{C}} \mathsf{d}(\mathbf{y}, \mathbf{c})$$

$$\boxed{\mathcal{D}_{\mathrm{MLD}} = \textit{nearest-codeword decoder}}$$

- True also for $\mathsf{QSC}(p)$ whenever $p < 1 - 1/q$

Error Correction

$$\mathbf{e} = [0\ldots0,\ \boxed{e_{i_1}},0\ldots0,\ \boxed{e_{i_2}},0\ldots0,\ \boxed{e_{i_t}},0\ldots0]$$



$i_1, i_2, \ldots, i_t$:	*error locations*	$e_{i_1}, e_{i_2}, \ldots, e_{i_t}$:	*error values* $(\neq 0)$

Full error correction: the task of recovering all $\{i_j\}$ and $\{e_{i_j}\}$ given $\mathbf{y}$

Theorem

Let $\mathcal{C}$ be an (n, M, d) code over F. There is a decoder $\mathcal{D} : F^n \to \mathcal{C}$ that recovers correctly every pattern of up to $\lfloor (d-1)/2 \rfloor$ errors for every channel $S = (F, F, \mathsf{Prob})$.

Linear Codes

- Assume $\mathbb{F}$ is a *finite field*
- $\mathcal{C} : (n, M, d)$ over $\mathbb{F}$ is called a *linear code* if $\mathcal{C}$ is a *linear sub-space* of $\mathbb{F}^n$ over $\mathbb{F}$
  - $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C},\ a_1, a_2 \in \mathbb{F} \quad \Rightarrow \quad a_1\mathbf{c}_1 + a_2\mathbf{c}_2 \in \mathcal{C}$
- A linear code $\mathcal{C}$ has $M = q^k$ codewords, where $k = \log_q M$ is the dimension of $\mathcal{C}$ as a linear space over $\mathbb{F}$
- $r = n - k$ is the redundancy of $\mathcal{C}$, $R = k/n$ its rate
- We use the notation $[n, k, d]$ to denote the parameters of a linear code
- A *generator* matrix for a linear code $\mathcal{C}$ is a $k \times n$ matrix G whose rows form a basis of $\mathcal{C}$.

Minimum Weight

- For an $[n, k, d]$ code $\mathcal{C}$,

$$\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C} \implies \mathbf{c}_1 - \mathbf{c}_2 \in \mathcal{C}\,, \text{ and } \mathsf{d}(\mathbf{c}_1, \mathbf{c}_2) = \mathsf{wt}(\mathbf{c}_1 - \mathbf{c}_2)\,.$$

  Therefore,

$$d = \min_{\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}\,:\,\mathbf{c}_1 \neq \mathbf{c}_2} \mathsf{d}(\mathbf{c}_1, \mathbf{c}_2) = \min_{\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{C}\,:\,\mathbf{c}_1 \neq \mathbf{c}_2} \mathsf{wt}(\mathbf{c}_1 - \mathbf{c}_2) = \min_{\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}} \mathsf{wt}(\mathbf{c})\,.$$

  $\Rightarrow$ *minimum distance is the same as minimum weight for linear codes*

  - Recall also that $\mathbf{0} \in \mathcal{C}$ and $\mathsf{d}(\mathbf{c}, \mathbf{0}) = \mathsf{wt}(\mathbf{c})$

Encoding Linear Codes

- Since $\mathrm{rank}(G) = k$, the map $\mathcal{E} : \mathbb{F}^k \to \mathcal{C}$ defined by

$$\mathcal{E} : \mathbf{u} \mapsto \mathbf{u}G$$

  is 1-1, and can serve as an encoding mechanism for $\mathcal{C}$.

- Applying elementary row operations and possibly reordering coordinates, we can bring G to the form

$$G = \left(\; I_k \; \middle| \; A \; \right) \qquad \textit{systematic generator matrix,}$$

  where I_k is a $k \times k$ identity matrix, and A is a $k \times (n-k)$ matrix.

$$\mathbf{u} \mapsto \mathbf{u}G = (\, \mathbf{u} \mid \mathbf{u}A \,) \qquad \textit{systematic encoding.}$$

- In a systematic encoding, the *information symbols* from $\mathbf{u}$ are transmitted 'as is,' and $n-k$ *check symbols* (or *redundancy* symbols, or *parity* symbols) are appended.

Parity Check Matrix

- Let $\mathcal{C} : [n, k, d]$. A *parity-check matrix (PCM)* of $\mathcal{C}$ is an $r \times n$ matrix H such that for all $\mathbf{c} \in \mathbb{F}^n$,

$$\mathbf{c} \in \mathcal{C} \qquad \Longleftrightarrow \qquad H\mathbf{c}^T = \mathbf{0}\,.$$

- For a generator matrix G of $\mathcal{C}$, we have

$$HG^T = 0 \Rightarrow GH^T = 0, \;\; \textbf{and} \;\; \dim\ker(G) = n - \mathrm{rank}(G) = n - k = r$$

- If $G = (\ I_k \mid A\ )$, then $H = (-A^T \mid I_{n-k}\ )$ is a (systematic) parity-check matrix.

Cosets and Syndromes

- Let $\mathbf{y} \in \mathbb{F}^n$. The *syndrome* of $\mathbf{y}$ (with respect to a PCM H of $\mathcal{C}$) is defined by

$$\mathbf{s} = H\mathbf{y}^T \in \mathbb{F}^{n-k}.$$

The set

$$\mathbf{y} + \mathcal{C} = \{\mathbf{y} + \mathbf{c} \ : \ \mathbf{c} \in \mathcal{C}\}$$

is a *coset* of $\mathcal{C}$ (as an additive subgroup) in $\mathbb{F}^n$.

- If $\mathbf{y}_1 \in \mathbf{y} + \mathcal{C}$, then

$$\mathbf{y}_1 - \mathbf{y} \in \mathcal{C} \implies H(\mathbf{y}_1 - \mathbf{y})^T = \mathbf{0} \implies H\mathbf{y}_1^T = H\mathbf{y}^T$$

$\implies$ *The syndrome is invariant for all* $\mathbf{y}_1 \in \mathbf{y} + \mathcal{C}$.

- Let $F = F_q$. Given a PCM H, there is a 1-1 correspondence between the q^{n-k} cosets of $\mathcal{C}$ in $\mathbb{F}^n$ and the q^{n-k} possible syndrome values (H is full-rank $\implies$ all values are attained).

Syndrome Decoding of Linear Codes

- $\mathbf{c} \in \mathcal{C}$ is sent and $\mathbf{y} = \mathbf{c} + \mathbf{e}$ is received on an additive channel
- $\mathbf{y}$ and $\mathbf{e}$ are in the same coset of $\mathcal{C}$
- Nearest-neighbor decoding of $\mathbf{y}$ calls for finding the closest codeword $\mathbf{c}$ to $\mathbf{y}$ $\implies$ find a vector $\mathbf{e}$ of *lowest weight* in $\mathbf{y}+\mathcal{C}$: a *coset leader*.
  - *coset leaders need not be unique* (when are they?)
- Decoding algorithm: upon receiving $\mathbf{y}$
  - compute the syndrome $\mathbf{s} = H\mathbf{y}^T$
  - find a coset leader $\mathbf{e}$ in the coset corresponding to $\mathbf{s}$
  - decode $\mathbf{y}$ into $\hat{\mathbf{c}} = \mathbf{y} - \mathbf{e}$
- If $n - k$ is (very) small, a table containing one leader per coset can be pre-computed. The table is indexed by $\mathbf{s}$.
- In general, however, syndrome decoding appears exponential in $n - k$. In fact, it has been shown to be NP-hard.

The Singleton Bound

- The *Singleton bound*.

Theorem (Singleton bound)

For any (n, M, d) *code over an alphabet of size* q,

$$d \leq n - (\log_q M) + 1 \ .$$

- Singleton bound for linear codes

Theorem (Singleton bound for linear codes)

For any linear $[n, k, d]$ *code over* $GF(q)$,

$$d \leq n - k + 1 \ .$$

- $\mathcal{C} : (n, M, d)$ (or, if linear, $\mathcal{C} : [n, k, d]$) is called *maximum distance separable (MDS)* if it meets the Singleton bound, namely $d = n - (\log_q M) + 1 \quad (d = n - k + 1)$.

The Sphere-Packing Bound

The *sphere* of center $\mathbf{c}$ and radius t in $\mathbb{F}_q^n$ is the set of vectors at Hamming distance t or less from $\mathbf{c}$. Its *volume* (cardinality) is

$$V_q(n,t) = \sum_{i=0}^{t} \binom{n}{i} (q-1)^i .$$

Theorem (The sphere-packing (SP) bound)

For any (n, M, d) code over $\mathbb{F}_q$,

$$M \cdot V_q(n, \lfloor (d-1)/2 \rfloor) \le q^n .$$

Proof. Spheres of radius $t = \lfloor (d-1)/2 \rfloor$ centered at codewords must be disjoint. $\square$

For a linear $[n,k,d]$ code, the bound becomes $V_q(n, \lfloor (d-1)/2 \rfloor) \le q^{n-k}$.
For $q = 2$,

$$\sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{n}{i} \le 2^{n-k}$$

The Gilbert-Varshamov bound

The Singleton and SP bounds set *necessary* conditions on the parameters of a code. The following is a *sufficient* condition:

Theorem (The Gilbert-Varshamov (GV) bound)

There exists an $[n, k, d]$ *code over the field* $\mathbb{F}_q$ *whenever*

$$V_q(n-1, d-2) < q^{n-k}.$$

Theorem

Let

$$\rho = \frac{q^k - 1}{q - 1} \cdot \frac{V_q(n, d-1)}{q^n}\ .$$

Then, a random $[n, k]$ *code has minimum distance* d *with* Prob $\geq 1 - \rho$.

Lots of codes are near the GV bound. But it's very hard to find them!

Asymptotic Bounds

- **Definition:** *relative distance* $\delta = d/n$
- We are interested in the behavior of δ and $R = (\log_q M)/n$ as $n \to \infty$.
- Singleton bound: $d \leq n - \lceil \log_q M \rceil + 1 \quad \Longrightarrow \quad \boxed{\; R \leq 1 - \delta + o(1)}$
- For the SP and GV bounds, we need estimates for $V_q(n,t)$
- **Definition:** *symmetric q-ary entropy function* $\mathsf{H}_q : [0,1] \to [0,1]$

$$\mathsf{H}_q(x) = -x \log_q x - (1-x) \log_q (1-x) + x \log_q (q-1) \;,$$

  - $\mathsf{H}_q(0) = 0,\ \mathsf{H}_q(1) = \log_q(q-1)$, strictly $\cap$-convex, $\max = 1$ at $x = 1 - 1/q$
  - coincides with $\mathsf{H}(x)$ when $q = 2$

Asymptotic Bounds (II)

Lemma. *For $0 \le t/n \le 1 - (1/q)$,*

$$V_q(n,t) = \sum_{i=0}^{t} \binom{n}{i} (q-1)^i \le q^{n\mathsf{H}_q(t/n)} .$$

Lemma. *For integers $0 \le t \le n$,*

$$V_q(n,t) \ge \binom{n}{t} (q-1)^t \ge \frac{1}{\sqrt{8t(1-(t/n))}} \cdot q^{n\mathsf{H}_q(t/n)} .$$

Theorem (Asymptotic SP bound)

For every $(n, q^{nR}, \delta n)$ code over $\mathbb{F}_q$,

$$R \le 1 - \mathsf{H}_q(\delta/2) + o(1) .$$

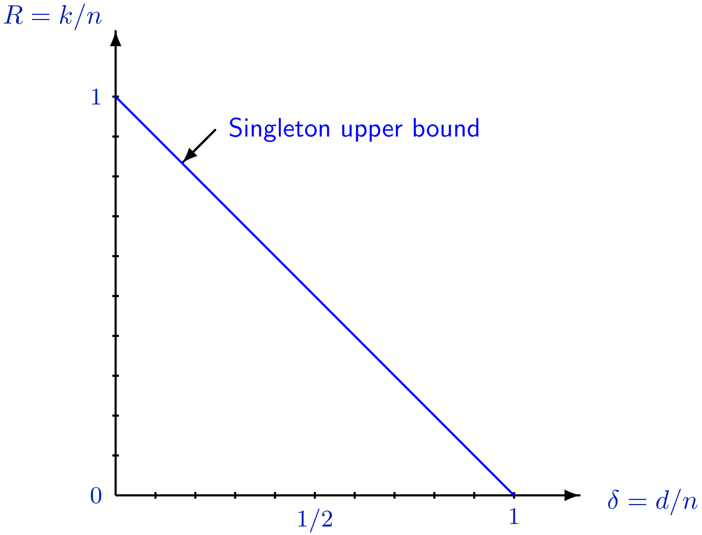
Theorem (Asymptotic GV bound)

Let n, nR, δn be positive integers such that $\delta \in (0, 1-(1/q)]$ and

$$R \le 1 - \mathsf{H}_q(\delta) .$$

Then, there exists a linear $[n, nR, \ge \delta n]$ code over Fq.

Plot of Asymptotic Bounds



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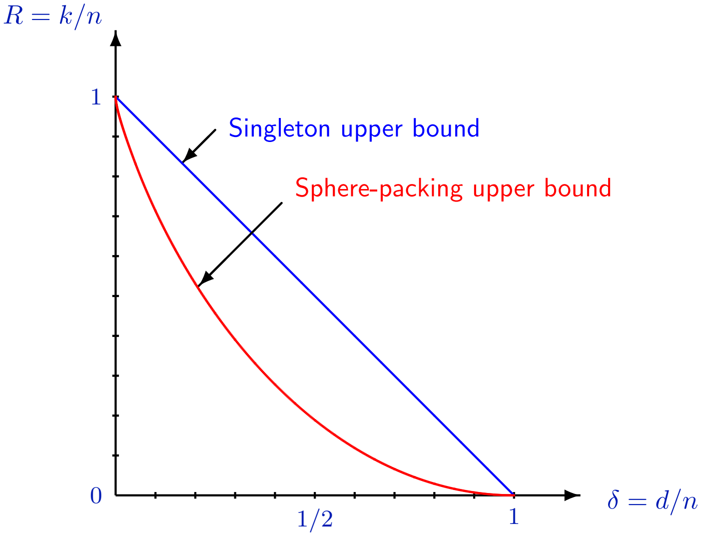

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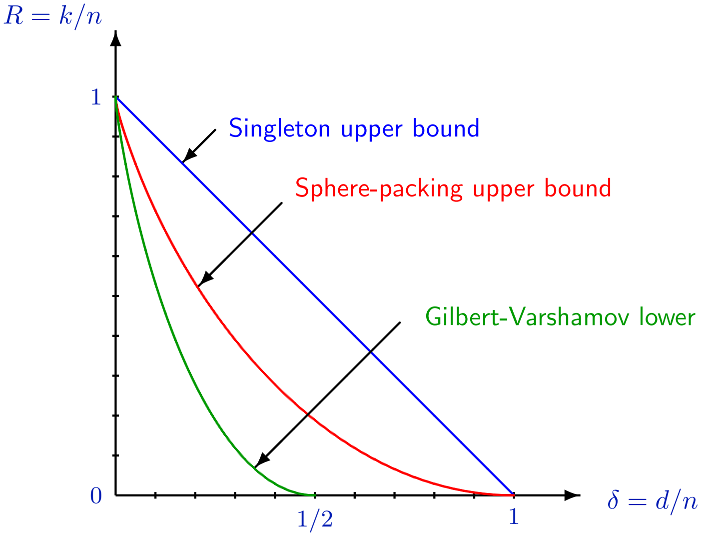

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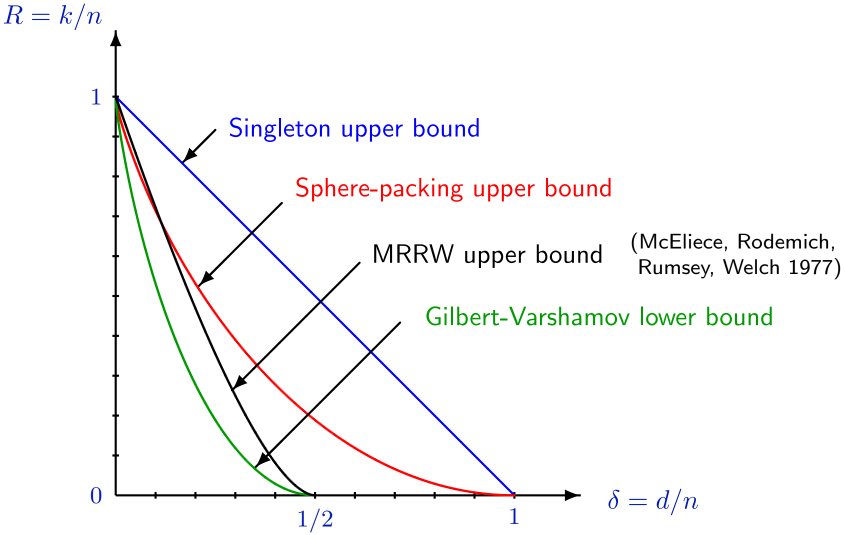

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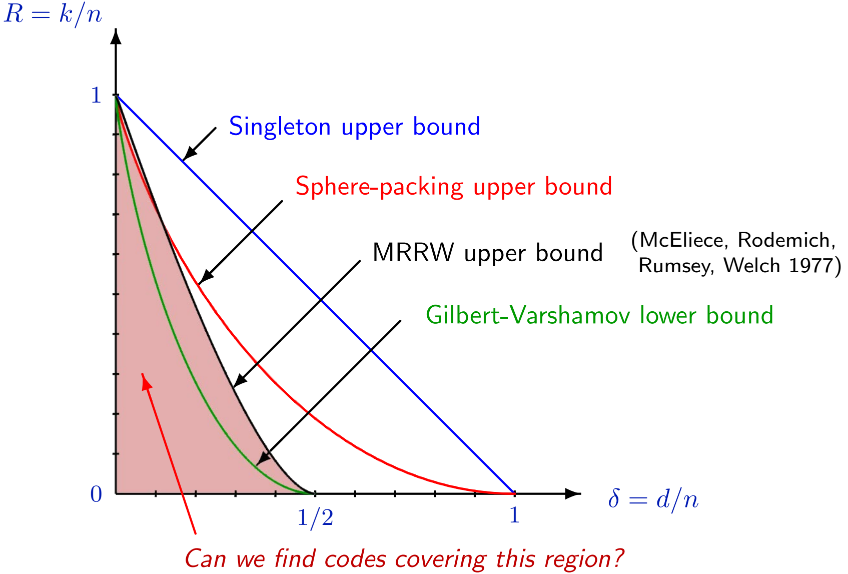

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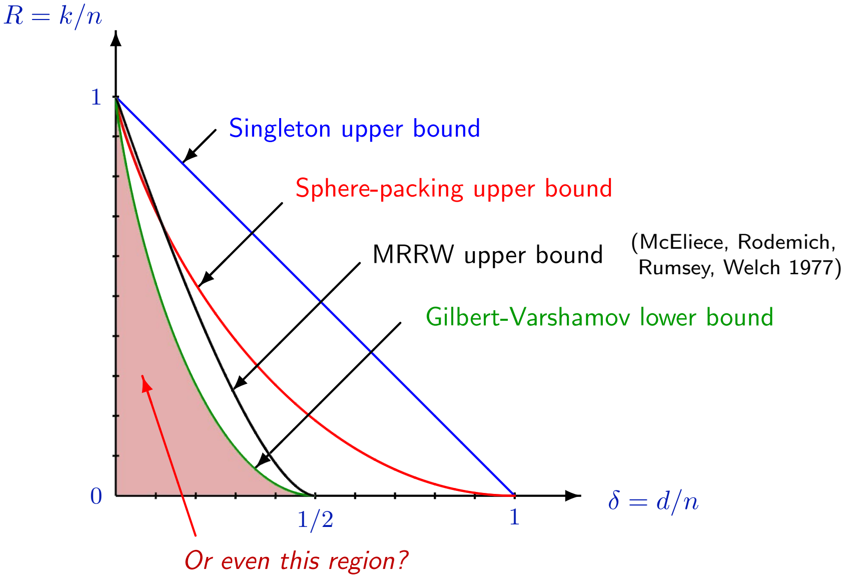

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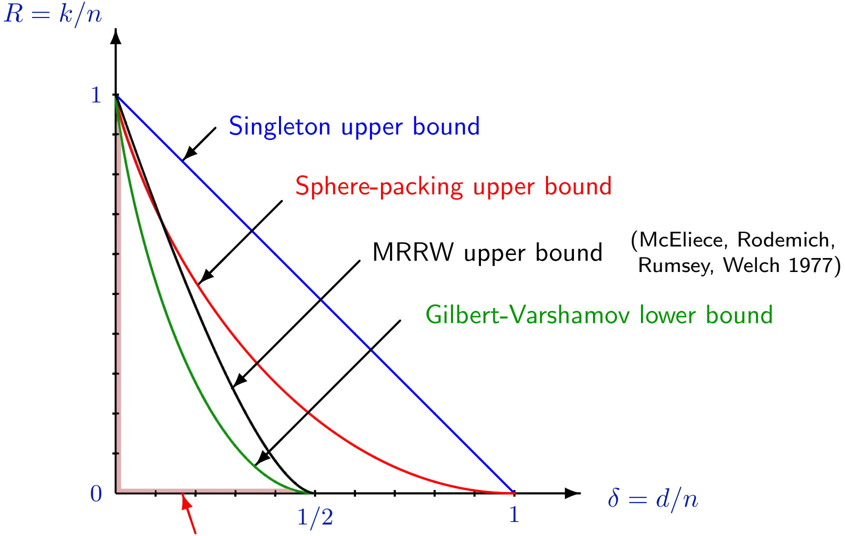

So far, we have only seen codes on these lines!

What we lose for decoding only up to (d-1)/2

- Hamming (sphere packing) bound

$$R \leq 1 - H(\delta/2) + o(1)$$

- Assume binary symmetric channel of parameter p.
  Channel capacity: $C = 1 - H(p)$
  $\Rightarrow$ with R arbitrarily close to $1 - H(p)$, can correct typical patterns of weight np with probability 1
  $\Rightarrow$ "equivalent minimum distance" $\approx 2np$
  $\Rightarrow$ $\delta \approx 2p$
  $\Rightarrow$ can achieve virtually zero-error communication with $R \approx 1 - H(\delta/2)$

Hamming bound curve

Generalized Reed-Solomon Codes

- Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, $n < q$, be distinct nonzero elements of $\mathbb{F}_q$, and let $v_1, v_2, \ldots, v_n$ be *nonzero* elements of $\mathbb{F}_q$ (not necessarily distinct). A *generalized Reed-Solomon (GRS)* code is a linear $[n, k, d]$ code $\mathcal{C}_{\mathrm{GRS}}$ with PCM

$$H_{\mathrm{GRS}} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{n-k-1} & \alpha_2^{n-k-1} & \ldots & \alpha_n^{n-k-1} \end{pmatrix} \begin{pmatrix} v_1 & & & \\ & v_2 & & 0 \\ 0 & & \ddots & \\ & & & v_n \end{pmatrix} .$$

α_j: *code locators* (distinct), v_j: *column multipliers* ($\neq 0$)

Generalized Reed-Solomon Codes

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α_j: *code locators* (distinct), v_j: *column multipliers* ($\neq 0$)

Theorem

$\mathcal{C}_{\mathrm{GRS}}$ is an MDS code, namely, $d = n - k + 1$.

Theorem

The dual of a GRS code is a GRS code.

GRS Encoding as Polynomial Evaluation

- For $\mathbf{u} = (u_0\, u_1\, \ldots\, u_{k-1})$, let $u(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_{k-1}x^{k-1}$. Then,

$$\mathbf{c} = \mathbf{u}G_{\text{GRS}} = \mathbf{u} \cdot \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_n^{k-1} \end{pmatrix} \begin{pmatrix} v'_1 & & & 0 \\ & v'_2 & & \\ 0 & & \ddots & \\ & & & v'_n \end{pmatrix}$$

$$= [\, v'_1 u(\alpha_1) \;\; v'_2 u(\alpha_2) \;\; \ldots \;\; v'_n u(\alpha_n)\,]$$

- Minimum distance now follows from the fact that a polynomial of degree $\leq k-1$ cannot have more than $k-1$ roots in $\mathbb{F}_q \implies \mathrm{wt}(\mathbf{c}) \geq n-k+1$.
- Decoding as *noisy interpolation*: reconstruct $u(x)$ from $(k+2t)$ noisy evaluations $u(\alpha_1)+e_1,\, u(\alpha_2)+e_2, \ldots, u(\alpha_{k+2t})+e_{k+2t}$, possible if at most t evaluations are corrupted.

Conventional Reed-Solomon Codes

- *Conventional Reed-Solomon (RS)* code: GRS code with $n|(q-1)$, $\alpha \in \mathbb{F}^*$ with $\mathcal{O}(\alpha) = n$,

$$\begin{array}{rcll} \alpha_j & = & \alpha^{j-1} , & 1 \le j \le n, \\ v_j & = & \alpha^{b(j-1)} , & 1 \le j \le n \,. \end{array}$$

- *Canonical PCM* of a RS code is given by

$$H_{\mathrm{RS}} = \begin{pmatrix} 1 & \alpha^b & \ldots & \alpha^{(n-1)b} \\ 1 & \alpha^{b+1} & \ldots & \alpha^{(n-1)(b+1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{b+d-2} & \ldots & \alpha^{(n-1)(b+d-2)} \end{pmatrix} \quad (\#\ \text{rows} = d{-}1 = n{-}k)$$

- $\mathbf{c} \in \mathcal{C}_{\mathrm{RS}} \iff H_{\mathrm{RS}}\mathbf{c}^T = \mathbf{0} \iff c(\alpha^\ell) = 0,\ \ell = b, b{+}1, \ldots, b{+}d{-}2$.
- $\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+d-2}$: *roots* of $\mathcal{C}_{\mathrm{RS}}$
- $g(x) = (x - \alpha^b)(x - \alpha^{b+1}) \cdots (x - \alpha^{b+d-2})$: *generator polynomial* of $\mathcal{C}_{\mathrm{RS}}$

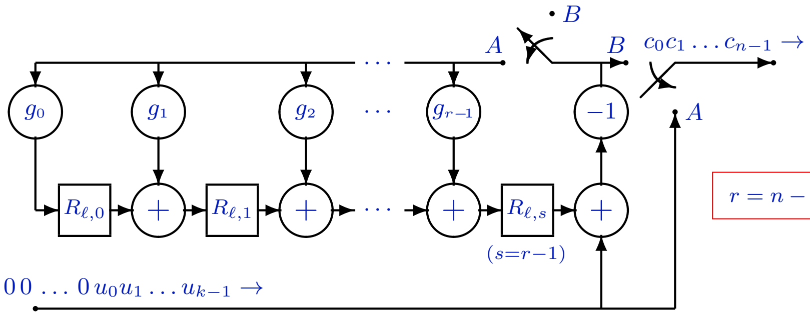
Systematic Encoding of RS Codes

- For $u(x) \in \mathbb{F}_q[x]_k$, let $r_u(x)$ be the unique polynomial in $\mathbb{F}_q[x]_{n-k}$ such that
$$r_u(x) \equiv x^{n-k}u(x) \mod g(x)$$

- Clearly, $x^{n-k}u(x) - r_u(x) \in \mathcal{C}_{\mathrm{RS}}$
- The mapping $\mathcal{E}_{\mathrm{RS}} : u(x) \mapsto x^{n-k}u(x) - r_u(x)$ is a *linear, systematic* encoding for $\mathcal{C}_{\mathrm{RS}}$

$$\begin{array}{rcccccccl}
[& u_{k-1} & u_{k-2} & \dots & u_0 & 0 & 0 & \dots & 0 \quad] \\
-[& 0 & 0 & \dots & 0 & r_{n-k-1} & r_{n-k-2} & \dots & r_0 \quad] \\
\hline
[& c_{n-1} & c_{n-2} & \dots & c_{n-k} & c_{n-k-1} & c_{n-k-2} & \dots & c_0 \quad]
\end{array}$$

Systematic Encoding Circuit



$\boxed{r = n - k}$

Switches:

- at A for k cycles
- at B for $r=n-k$ cycles

Register contents:

$$R_\ell(x) = \sum_{i=0}^{r-1} R_{\ell,i} x^i, \quad 0 \le \ell < k$$

with initial condition

$$R_0(x) = 0$$

Decoding Generalized Reed-Solomon Codes

- We consider $\mathcal{C}_{\mathrm{GRS}}$ over $\mathbb{F}_q$ with PCM

$$H_{\mathrm{GRS}} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{d-2} & \alpha_2^{d-2} & \dots & \alpha_n^{d-2} \end{pmatrix} \begin{pmatrix} v_1 & & & 0 \\ & v_2 & & \\ 0 & & \ddots & \\ & & & v_n \end{pmatrix}$$

  with $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}_q^*$ distinct, and $v_1, v_2, \dots, v_n \in \mathbb{F}_q^*$
- Codeword $\mathbf{c}$ transmitted, word $\mathbf{y}$ received, with error vector

$$\mathbf{e} = (e_1 \; e_2 \; \dots \; e_n) = \mathbf{y} - \mathbf{c}$$

- $J = \{\kappa \; : \; e_\kappa \neq 0\}$ set of *error locations*
- We describe an algorithm that correctly decodes $\mathbf{y}$ to $\mathbf{c}$, under the assumption $|J| \leq \frac{1}{2}(d-1)$.

Syndrome Computation

- First step of the decoding algorithm

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{d-2} \end{pmatrix} = H_{\mathrm{GRS}} \mathbf{y}^T = H_{\mathrm{GRS}} \mathbf{e}^T$$

$$S_\ell = \sum_{j=1}^{n} y_j v_j \alpha_j^\ell = \sum_{j=1}^{n} e_j v_j \alpha_j^\ell = \sum_{j \in J} e_j v_j \alpha_j^\ell \,, \quad \ell = 0, 1, \ldots, d-2$$

Example. For RS codes, we have $\alpha_j = \alpha^{j-1}$ and $v_j = \alpha^{b(j-1)}$, so

$$S_\ell = \sum_{j=1}^{n} y_j \alpha^{(j-1)(b+\ell)} = y(\alpha^{b+\ell}) \,, \quad \ell = 0, 1, \ldots, d-2 \,.$$

- *Syndrome polynomial*:

$$S(x) = \sum_{\ell=0}^{d-2} S_\ell x^\ell = \sum_{\ell=0}^{d-2} x^\ell \sum_{j \in J} e_j v_j \alpha_j^\ell = \sum_{j \in J} e_j v_j \sum_{\ell=0}^{d-2} (\alpha_j x)^\ell \,.$$

A Congruence for the Syndrome Polynomial

$$S(x) = \sum_{j\in J} e_j v_j \sum_{\ell=0}^{d-2} (\alpha_j x)^\ell \,.$$

- We have

$$\sum_{\ell=0}^{d-2} (\alpha_j x)^\ell \equiv \frac{1}{1-\alpha_j x} \pmod{x^{d-1}}$$

$$\Longrightarrow \quad \boxed{S(x) \equiv \sum_{j\in J} \frac{e_j v_j}{1-\alpha_j x} \pmod{x^{d-1}}} \qquad \left(\textstyle\sum_\phi \Box = 0\right)$$

More Auxiliary Polynomials

- *Error locator polynomial (ELP)*

$$\Lambda(x) = \prod_{j \in J} (1 - \alpha_j x) \qquad \left(\prod_{\phi} \square \triangleq 1 \right)$$

- *Error evaluator polynomial (EEP)*

$$\Gamma(x) = \sum_{j \in J} e_j v_j \prod_{m \in J \setminus \{j\}} (1 - \alpha_m x)$$

- $\Lambda(\alpha_\kappa^{-1}) = 0 \quad \Longleftrightarrow \quad \kappa \in J$ *roots of EEP point to error locations*
- $\Gamma(\alpha_\kappa^{-1}) = e_\kappa v_\kappa \prod_{m \in J \setminus \{\kappa\}} (1 - \alpha_m \alpha_\kappa^{-1}) \neq 0$

$$\Longrightarrow \quad \boxed{\gcd(\Lambda(x), \Gamma(x)) = 1}$$

- The degrees of ELP and EEP satisfy

$$\deg \Lambda = |J| \quad \text{and} \quad \deg \Gamma < |J|$$

Of course, we don't know $\Lambda(x)$, $\Gamma(x)$: our goal is to find them

Key Equation of GRS Decoding

Since $|J| \leq \frac{1}{2}(d-1)$, we have

(1) $\boxed{\deg \Lambda \leq \frac{1}{2}(d-1)}$ and (2) $\boxed{\deg \Gamma < \frac{1}{2}(d-1)}$

The ELP and the EEP are related by

$$\Gamma(x) = \sum_{j \in J} e_j v_j \prod_{m \in J \setminus \{j\}} (1 - \alpha_m x) = \sum_{j \in J} e_j v_j \frac{\Lambda(x)}{1 - \alpha_j x} = \Lambda(x) \sum_{j \in J} \frac{e_j v_j}{1 - \alpha_j x}$$

$$\Longrightarrow (3) \quad \boxed{\Lambda(x) S(x) \equiv \Gamma(x) \pmod{x^{d-1}}}$$

(1)+(2)+(3): *key equation of GRS decoding*

- (3) is a set of $d-1$ *linear* equations in the coefficients of Λ and Γ
- $\lfloor \frac{1}{2}(d-1) \rfloor$ equations depend only on Λ (corresponding to x^i, $i \geq \frac{1}{2}(d-1)$)
- we can solve for Λ, find its root set J, then solve *linear* equations for e_j
- straightforward solution leads to $O(d^3)$ algorithm — we'll present an $O(d^2)$ one

Solving the Key Equation

- Apply the Euclidean algorithm with
  $a(x) = x^{d-1}$ and $b(x) = S(x)$, to produce
  $\Lambda(x) = c \cdot t_h(x)$ and $\Gamma(x) = c \cdot r_h(x)$
  [the key equation guarantees conditions (C1)–(C3)].
  How do we find h —the stopping index?

Theorem

The solution to the key equation is unique up to a scalar constant, and it is obtained with the Euclidean algorithm by stopping at the unique index h such that

$$\deg r_h < \tfrac{1}{2}(d-1) \leq \deg r_{h-1}$$

Finding the Error Values

- *Formal derivatives* in finite fields: $\left[\sum_{i=0}^{s} a_i x^i\right]' = \sum_{i=1}^{s} i a_i x^{i-1}$
  $(a(x)b(x))' = a'(x)b(x) + a(x)b'(x)$ (not surprising)
- For the ELP, we have

$$\Lambda(x) = \prod_{j \in J} (1 - \alpha_j x) \quad \Longrightarrow \quad \Lambda'(x) = \sum_{j \in J} (-\alpha_j) \prod_{m \in J \setminus \{j\}} (1 - \alpha_m x),$$

and, for $\kappa \in J$,

$$\Lambda'(\alpha_\kappa^{-1}) = -\alpha_\kappa \prod_{m \in J \setminus \{\kappa\}} (1 - \alpha_m \alpha_\kappa^{-1}),$$

$$\Gamma(\alpha_\kappa^{-1}) = e_\kappa v_\kappa \prod_{m \in J \setminus \{\kappa\}} (1 - \alpha_m \alpha_\kappa^{-1})$$

- Therefore, for all error locations $\kappa \in J$, we obtain

$$\boxed{e_\kappa = -\frac{\alpha_\kappa}{v_\kappa} \cdot \frac{\Gamma(\alpha_\kappa^{-1})}{\Lambda'(\alpha_\kappa^{-1})}}$$

Forney's algorithm for error values

Summary of GRS Decoding

Input: received word $(y_1\ y_2\ \ldots\ y_n) \in \mathbb{F}_q^n$.
Output: error vector $(e_1\ e_2\ \ldots\ e_n) \in \mathbb{F}_q^n$.

1. *Syndrome computation:* Compute the polynomial $S(x) = \sum_{\ell=0}^{d-2} S_\ell x^\ell$ by
$$S_\ell = \sum_{j=1}^{n} y_j v_j \alpha_j^\ell\ , \quad \ell = 0, 1, \ldots, d-2\ .$$
2. *Solving the key equation:* Apply Euclid's algorithm to $a(x) \leftarrow x^{d-1}$ and $b(x) \leftarrow S(x)$ to produce $\Lambda(x) \leftarrow t_h(x)$ and $\Gamma(x) \leftarrow r_h(x)$, where h is the smallest index i for which $\deg r_i < \frac{1}{2}(d-1)$.
3. *Forney's algorithm:* Compute the error locations and values by
$$e_j = \begin{cases} -\dfrac{\alpha_j}{v_j} \cdot \dfrac{\Gamma(\alpha_j^{-1})}{\Lambda'(\alpha_j^{-1})} & \text{if } \Lambda(\alpha_j^{-1}) = 0 \\ 0 & \text{otherwise} \end{cases}\ , \quad j = 1, 2, \ldots, n\ .$$

Complexity: 1. $O(dn)$ 2. $O\big((|J|+1)\, d\big)$ 3. $O\big((|J|+1)n\big)$

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