$$\rightarrow (x^{\alpha})' = \alpha \cdot x^{\alpha - 1} \qquad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

6.3.1.

d) $$\left(x\sqrt{1+x^2}\right)' = 1 \cdot \sqrt{1+x^2} + x \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$= \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}$$

g) $$\left(\sin^3(x)\right)' = 3\sin^2(x) \cdot \cos(x)$$

j) $$\left(\sin\left(\frac{1}{x} + x^2\right)\right)' = \cos\left(\frac{1}{x} + x^2\right) \cdot \left(-\frac{1}{x^2} + 2x\right)$$

m) $$\left(\cos(x) - 1\right)^{100\,\prime} = 100 \cdot (\cos(x) - 1)^{99} \cdot (-\sin(x))$$

3. $$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\cos(x)^2 + \sin(x)^2 = 1$$

$$\cosh(x)^2 - \sinh(x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 =$$

$$\frac{\cancel{e^{2x}} + \cancel{e^{-2x}} + 2}{4} - \frac{\cancel{e^{2x}} + \cancel{e^{-2x}} - 2}{4} = \frac{2}{4} + \frac{2}{4} = 1$$

$$\sinh(x)' = \left(\frac{e^x - e^{-x}}{2}\right)' = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x + e^{-x}}{2}$$

$$= \cosh(x)$$

---

6. a) $$\left(\left(\frac{1+x^3}{1-x^3}\right)^{\frac{1}{3}}\right)' = \frac{1}{3}\left(\frac{1+x^3}{1-x^3}\right)^{-\frac{2}{3}} \cdot \frac{3x^2(1-x^3) - (1+x^3)(-3x^2)}{(1-x^3)^2}$$

$$= \frac{1}{3}\left(\frac{1-x^3}{1+x^3}\right)^{\frac{2}{3}} \cdot \frac{3x^2(1-x^3) + 3(1+x^3)x^2}{(1-x^3)^2} = \frac{1}{3} \frac{1}{(1+x^3)} \cdot \frac{6x^2}{(1-x^3)^{2-\frac{2}{3}}}$$

$$= \frac{2x^2}{(1+x^3)(1-x^3)^{4/3}}$$

6. b) $\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right)' = \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} \cdot$

$$\cdot\left(1+\frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\left(1+\frac{1}{2\sqrt{x}}\right)\right)$$

8. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{si } x = 0 \\ \underbrace{x^2 \cdot \sin\left(\frac{1}{x}\right)} & \text{si } x \neq 0 \end{cases}$$

$a = 0$

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} =$$

$$= \lim_{x\to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0 \to f'(0) = 0$$

$x \neq 0$, $f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$

$$= 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$\to f'(x) = \begin{cases} 0 & \text{si } x = 0 \\ \underbrace{2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)}_{\to 0} & \text{si } x \neq 0 \end{cases}$$

$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ no existe

9. a) $\rightarrow$ $f^{(0)} = f$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$\rightarrow f^{(4)}(x) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

$\vdots$

$$f^{(n)}(x) = \begin{cases} \sin(x) & n = 4\cdot k \\ \cos(x) & n = 4k+1 \\ -\sin(x) & n = 4k+2 \\ -\cos(x) & n = 4k+3 \end{cases}$$

c) $f(x) = e^x$, $\quad f^{(n)}(x) = e^x$

d) $f(x) = \log(x)$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f'''(x) = 2\frac{1}{x^3}$

$f^{(4)}(x) = -2\cdot 3\frac{1}{x^4}$

$f^{(5)} = 2\cdot 3\cdot 4\cdot \frac{1}{x^5}$

$n \geq 1 \qquad f^{(n)}(x) = (-1)^{n+1}(n-1)!\,\frac{1}{x^n}$

e) $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$

$f'(x) = -(1-x)^{-2} \cdot (-1) = (1-x)^{-2}$

$f''(x) = -2\,(1-x)^{-3} \cdot (-1) = 2\,(1-x)^{-3}$

$f'''(x) = 2 \cdot 3\,(1-x)^{-4}$

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Regla de l'Hôpital

f, g derivables en un entorno de a

$g \neq 0$ en un entorno de a

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\infty}{\infty}$$

$a \in \mathbb{R} \cup \{\pm\infty\}$

Si existe $\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ $\in \mathbb{R}$, entonces existe

$$\Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

Ejemplo práctico:

$$\lim_{x \to +\infty} \frac{\log(x)}{x} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0$$

$$6.4.1\ a)\quad \lim_{x\to 0} \frac{1-e^{4x}}{\sin(x)} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{1-e^{4x}}{\sin(x)} = \lim_{x\to 0} \frac{-4e^{4x}}{\cos(x)} = \frac{-4}{1} = -4$$

$$b)\quad \lim_{x\to 1} \frac{\log(x)}{\cos(\pi x)+1} \overset{\nearrow 0/0}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{-\sin(\pi x)\cdot \pi} = \frac{1}{0} = \infty$$

$$\| $$

$$\frac{1}{\lim_{x\to 1} \frac{\cos(\pi x)+1}{\log(x)}} = \frac{1}{\lim_{x\to 1} \frac{-\sin(\pi x)\cdot\pi}{\frac{1}{x}}} = \frac{1}{0} = \infty$$

$$\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{\lim_{x\to a} f(x)} \quad \text{si } \exists \lim f(x)$$

$$c)\quad \lim_{x\to 0} \frac{e^x - \cos(x)}{\sin(\pi x)} \underset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x + \sin(x)}{\cos(\pi x)\cdot \pi} = \frac{1}{\pi}$$

$$2a)\quad \lim_{x\to \frac{\pi}{2}^+} \frac{\log(\sin(x))}{\cos(x)} \overset{\nearrow 0/0}{=} \lim_{x\to\frac{\pi}{2}^+} \frac{\frac{1}{\sin(x)}\cdot\cos(x)}{-\sin(x)} =$$

$$= \lim_{x\to\frac{\pi}{2}^+} -\frac{\cos(x)}{\sin(x)^2} = 0$$

b) $\lim\limits_{x\to 1} \dfrac{\sin(\log(x))}{\sin(\pi x)} \overset{\nearrow 0/0}{=} \lim\limits_{x\to 1} \dfrac{\cos(\log(x))\cdot \frac{1}{x}}{\cos(\pi x)\cdot \pi}$

$= -\dfrac{1}{\pi}$

3) a) $\lim\limits_{x\to +\infty} \dfrac{x^2}{e^x} \overset{\nearrow \infty/\infty}{=} \lim\limits_{x\to +\infty} \dfrac{2x}{e^x} \overset{\nearrow \infty/\infty}{=} \lim\limits_{x\to +\infty} \dfrac{2}{e^x} = 0$

d) $\lim\limits_{x\to +\infty} e^{\sqrt{x+1}} - e^{\sqrt{x}} \overset{\nearrow \infty - \infty}{=}$

$\lim\limits_{x\to +\infty} e^{\sqrt{x}} \left(e^{\sqrt{x+1}-\sqrt{x}} - 1 \right) =$

$\infty \cdot 0$

$= \lim\limits_{x\to +\infty} e^{\sqrt{x}} \left(e^{\frac{1}{\sqrt{x+1}+\sqrt{x}}} - 1 \right) \overset{\nearrow}{=}$

$= \lim\limits_{x\to +\infty} \dfrac{e^{\frac{1}{\sqrt{x+1}+\sqrt{x}}} - 1}{e^{-\sqrt{x}}} \overset{\nearrow 0/0}{=}$

L'Hôpital

$= \lim\limits_{x\to +\infty} \dfrac{e^{\frac{1}{\sqrt{x+1}+\sqrt{x}}} \cdot \left(-\frac{1}{(\sqrt{x+1}+\sqrt{x})^2} \right) \cdot \left(\frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x}} \right)}{e^{-\sqrt{x}} \cdot \left(-\frac{1}{2\sqrt{x}} \right)} = \infty$

$1 \leftarrow$

$0 \quad 0$

$0 \quad 0$

$$\sim e^{\sqrt{x}}\left(\sqrt{x+1}-\sqrt{x}\right) \qquad e^x - 1 \sim x$$

$$e^{\sqrt{x}}\,\frac{1}{\sqrt{x+1}+\sqrt{x}} \to +\infty$$

Otra solución: (no fácil)

$$\lim_{x\to+\infty} e^{\sqrt{x+1}} - e^{\sqrt{x}} = \lim_{x\to+\infty} \frac{e^{\sqrt{x+1}-\sqrt{x}}-1}{e^{-\sqrt{x}}} \underset{\text{l'Hôpital}}{=}$$

$$= \lim_{x\to+\infty} \frac{e^{\sqrt{x+1}-\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x+1}}-\frac{1}{2\sqrt{x}}\right)}{e^{-\sqrt{x}}\left(-\frac{1}{2\sqrt{x}}\right)} \underset{\text{Simplificando}}{=} \lim e^{\sqrt{x+1}}\cdot\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}}$$

conjugado

$$= \lim e^{\sqrt{x+1}}\,\frac{1}{\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x}\right)} \underset{\text{órdenes}}{=} +\infty$$