6.1.9. a) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} ax+b & \text{si } x \leq 1 \\ \frac{1}{x} & \text{si } x > 1 \end{cases}$$

Determinar $a$ y $b$ para que $f$ sea derivable.

$ax+b$ es derivable $\to$ $x<1$ $f$ es derivable

$\frac{1}{x}$ es derivable para $x \neq 0$ $\to$ $x>1$ $f$ es derivable

Hay que ver si $f$ es derivable en $1$, es decir, si existe

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}.$$

- $f(1) = a+b$

- $\lim_{x\to 1^+} \dfrac{\frac{1}{x}-(a+b)}{x-1} = \dfrac{1-(a+b)}{0} = \pm\infty$ ← Si $a+b \neq 1$

$\Downarrow$

$f$ no es derivable en $1$

- $\boxed{\lim_{x\to 1^-} \dfrac{ax+b-(a+b)}{x-1}}$

Necesitamos que $a+b=1$.

$$\lim_{x\to 1^+} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1^+} \frac{\left(\frac{1-x}{x}\right)}{x-1} = \lim_{x\to 1^+} -\frac{1}{x} = -1$$

$$\lim_{x\to 1} \frac{a(x-1)}{x-1} = a$$

Necesito $a+b=1$ y $a=-1$ $\Rightarrow$ $a=-1$ y $b=2$

Alternativa

$$\lim_{x\to 1^-} \frac{\overbrace{f(x)}^{g(x)} - \overbrace{f(1)}^{g(1)}}{x-1} = \underbrace{g'(1)}_{a} \quad \text{donde } g(x) = ax+b$$

$$\lim_{x\to 1^+} \frac{\overbrace{f(x)}^{h(x)} - f(1)}{x-1} = h'(1) \quad \text{donde } h(x) = \frac{1}{x}$$

si $\underbrace{f(1) = h(1)}$ $\Leftrightarrow$ $g(1) = h(1)$

$f$ derivable en 1 $\Leftrightarrow$ $\begin{cases} ① \ f \text{ es continua} \\ ② \ g'(1) = h'(1) \end{cases}$

$g(x) = ax+b$, $g'(x) = a$, $g'(1) = a$

$h(x) = \frac{1}{x}$, $h'(x) = -\frac{1}{x^2}$, $h'(1) = -1$

① $a + b = 1$

② $a = -1$

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6) $f(x) = \begin{cases} 2x+2 & \text{si } x \leq a \\ 2x^2 & \text{si } x > a \end{cases}$

① $f$ continua $\Leftrightarrow$ $2a+2 = 2a^2$ $\Leftrightarrow$ $a^2 = a+1$

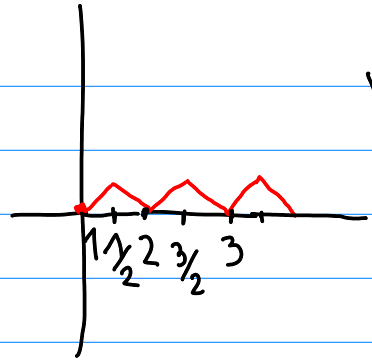
$\Leftrightarrow$ $a = \frac{1 \pm \sqrt{5}}{2}$

② $g'(a) = h'(a)$, donde $g(x) = 2x+2$, $h(x) = 2x^2$

$\Updownarrow$ $\quad g'(x) = 2$, $h'(x) = 4x$

$2 = 4a$ $\Leftrightarrow$ $a = \frac{1}{2}$

No es derivable.




$[x]$ no es derivable en $\frac{1}{2}\mathbb{Z}$

$$\frac{1}{2}\mathbb{Z} = \left\{ \dots, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \right\}$$

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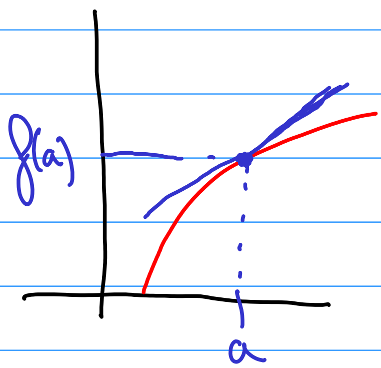
$f$ derivable en $a$

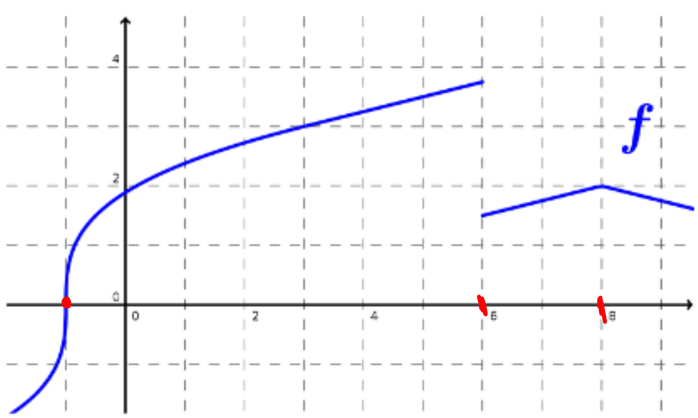
Recta tangente a $f$ en el punto $(a, f(a))$ es

$$r(x) = f(a) + f'(a)(x-a)$$




$$\lim_{x \to a} \frac{f(x) - r(x)}{x-a} = 0$$

1. (*) A partir del gráfico determinar en qué puntos no es derivable la funcion $f$



4. a) $f(x) = x^2$, $p = (3,9)$, $\overset{a=3}{f(3) = 9}$

$f'(x) = 2x$, $f'(3) = 6$

$r(x) = 9 + 6(x-3) = 6x - 9$

5. $a, b, c \in \mathbb{R}$

$f(x) = x^2 + ax + b$

$g(x) = cx - x^2$

Quiero que las rectas tangentes a $f$ y $g$ en $(3,3)$ coincidan.

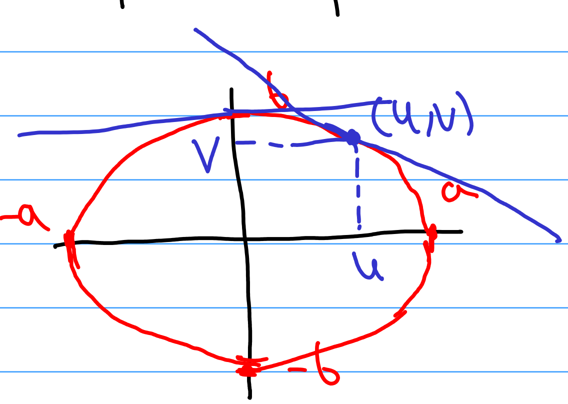
$$\Leftrightarrow \begin{cases} f(3) = 3 \\ g(3) = 3 \\ f'(3) = g'(3) \end{cases} \Leftrightarrow \begin{cases} 9 + 3a + b = 3 \\ 3c - 9 = 3 \\ 6 + a = c - 6 \end{cases}$$

$f'(x) = 2x + a$, $g'(x) = c - 2x$

$$\Leftrightarrow \begin{cases} c = 4 \\ a = -8 \\ 9 - 24 + b = 3 \end{cases} \Leftrightarrow \begin{cases} a = -8 \\ b = 18 \\ c = 4 \end{cases}$$

8. $\rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\rightarrow \dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$



$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}, \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$$

Puedo considerar $f_+ : [-a, a] \to \mathbb{R}$

$$f_+(x) = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$f_- : [-a, a] \to \mathbb{R}, \quad f_-(x) = -b\sqrt{1 - \frac{x^2}{a^2}}$$

Caso $v > 0$ $\quad f_+(u) = b\sqrt{1 - \frac{u^2}{a^2}} = v$

$$f_+'(x) = b\,\frac{1}{2\sqrt{1 - \frac{x^2}{a^2}}}\cdot\left(-\frac{2x}{a^2}\right) \qquad (\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$(f \circ g)'(x) = f'(g(x))\cdot g'(x)$$

$$f_+'(u) = -\frac{b}{a^2}\,\frac{u}{\sqrt{1 - \frac{u^2}{a^2}}} = \underbrace{-\frac{b^2}{a^2}\,\frac{u}{v}}$$

Recta tangente por $(u, v)$

$$\boxed{y} = v - \frac{b^2}{a^2}\cdot\frac{u}{v}\cdot(x - u) = -\frac{b^2}{a^2}\cdot\frac{u}{v}\cdot x + v + \frac{b^2}{a^2}\frac{u^2}{v}$$

$$= -\frac{b^2}{a^2}\cdot\frac{u}{v}\cdot x + \frac{b^2}{v}\left(\frac{v^2}{b^2} + \frac{u^2}{a^2}\right) = \boxed{-\frac{b^2}{a^2}\cdot\frac{u}{v}\cdot x + \frac{b^2}{v}}$$

Ejemplo: si $(u, v) = (0, b) \to y = b$

Caso $v < 0 \to$ uso $f_-$

Caso $v = 0$

$(u, v) = (a, 0) \to x = a$

$(u, v) = (-a, 0) \to x = -a$

6.3. $(x^\alpha)' = \alpha x^{\alpha-1}$, $\sin(x)' = \cos(x)$

$\cos(x)' = -\sin(x)$

$(\log(x))' = \frac{1}{x}$, $x>0$, $(e^x)' = e^x$

$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$\left(\frac{1}{f}\right)'(x) = \left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f(x)^2}$$

$$\left(\frac{f(x)}{g(x)}\right)'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}.$$

6.3.1 a) $\left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}\right)' = \left(x^{-1} + 2x^{-2} + 3x^{-3}\right)' =$

$= -1 \cdot x^{-2} + 2 \cdot (-2)\, x^{-3} + 3(-3)\, x^{-4} =$

$= \frac{-1}{x^2} - \frac{4}{x^3} - \frac{9}{x^4}$

b) $\left(\frac{ax+b}{cx+d}\right)' = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2} =$

$= \frac{ad-bc}{(cx+d)^2}$