5. Sea $f, g \in \mathcal{H}(D)$ siendo D es el disco unitario y tales que no se anulan. Probar que si $\frac{f'}{f}(\frac{1}{n}) = \frac{g'}{g}(\frac{1}{n})\ \forall n$ entonces existe $k \in \mathbb{C}$ tal que $f(z) = kg(z)\ \forall z$.

$\{\frac{1}{n} : n \in \mathbb{Z}^+\}$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} = 0 \Leftrightarrow f'g - fg' = 0 \Leftrightarrow f'g = fg' \Leftrightarrow$$

$$\frac{f'}{f} = \frac{g'}{g}$$

a es polo de orden n de f

$$\operatorname{Res}(f, a) = \lim_{z \to a} \frac{\partial^{n-1}}{\partial z^{n-1}} \frac{1}{(n-1)!} (z-a)^n f(z)$$

$g \in H(\Omega)$

$$f(z) = \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + g(z)$$

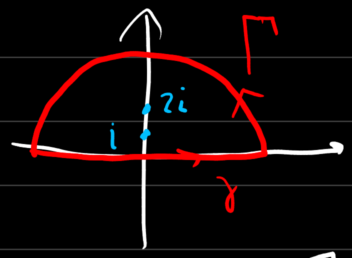
$$(z-a)^3 f(z) = \cancel{a_{-3}} + \cancel{a_{-2}(z-a)} + a_{-1}(z-a)^2 + (z-a)^3 g(z)$$

$$\frac{\partial^2}{\partial z^2} (z-a)^3 f(z) = 2!\, a_{-1} + \left((z-a)^3 g(z)\right)'' \xrightarrow[z \to a]{} 2!\, a_{-1}$$

$$\lim_{z \to a} (z-a)^n f(z)$$

(e) $\displaystyle\int_0^{+\infty} \frac{\cos ax}{(x^2+1)(x^2+4)}\,dx$ $\quad a>0$ $\qquad (x-i)(x+i)(x-2i)(x+2i)$

$$\frac{1}{2}\int_{-\infty}^{+\infty} dx$$



$\gamma_R : [-R, R] \to \mathbb{C} \;/\; \gamma_R(t) = t$

$\Gamma_R : [0, \pi] \to \mathbb{C} \;/\; \Gamma_R(t) = Re^{it}$

$\alpha_R = \gamma_R \cdot \Gamma_R$

$$\oint_{\alpha_R} \frac{e^{iaz}}{(z^2+1)(z^2+4)}\,dz \qquad e^{iaz} = \cos az + i\sin az$$

Lema 2 ***Lema de Jordan.***
Si $f(z)$ es una función compleja continua para todo z tal que $|z| \geq R_0$, que cumple $\lim_{z\to\infty} f(z) = 0$ entonces:

$$\lim_{R\to+\infty} \int_{\Gamma_R} e^{isz} f(z)dz = 0$$

donde $s > 0$ es constante y Γ_R es un arco contenido en la semicircunferencia: $z = Re^{it}, t \in [0, \pi]$.

$$\lim_{z\to\infty} \frac{1}{(z^2+1)(z^2+4)} = 0 \quad ✓$$

$$\oint_{\alpha_R} \underbrace{\frac{e^{iaz}}{(z^2+1)(z^2+4)}}_{g(z)}\,dz = \int_{\gamma_R} g\,dz + \underbrace{\int_{\Gamma_R} g\,dz}_{\xrightarrow{R\to+\infty} 0}$$

$$I_R = \int_{-R}^{R} g(t)\cdot 1\,dt = \int_{-R}^{R} \frac{\cos at}{(t^2+1)(t^2+4)}\,dt + i\int_{-R}^{R} \frac{\sin at}{(t^2+1)(t^2+4)}\,dt$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)(x^2+4)}\,dx = \operatorname{Re}\left(\lim_{R\to+\infty} I_R\right)$$

g tiene polos de orden 1 en $i, 2i$
es meromorfa en $\{z\in\mathbb{C} : \operatorname{Im}(z) > -1/2\}$

$$\oint_{\alpha_R} g(z)\,dz = 2\pi i\left(\operatorname{Res}(g, i) + \operatorname{Res}(g, 2i)\right)$$

$R > 3$

$$\mathrm{Res}(g, i) = \lim_{z\to i} (z-i)\, g(z) = \lim_{z\to i} \frac{e^{iaz}}{(z+i)(z^2+4)}$$

$$= \frac{e^{-a}}{2i \cdot 3} = \frac{e^{-a}}{6i}$$

$$\mathrm{Res}(g, 2i) = \lim_{z\to 2i} (z-2i)\, g(z) = \lim_{z\to 2i} \frac{e^{iaz}}{(z^2+1)(z+2i)}$$

$$= \frac{e^{-2a}}{-3 \cdot 4i} = \frac{-e^{-2a}}{12i}$$

$$\Rightarrow \int_{\gamma_R} g\, dz = 2\pi \cancel{i} \left(\frac{e^{-a}}{6\cancel{i}} - \frac{e^{-2a}}{12\cancel{i}} \right) = \pi \left[\frac{e^{-a}}{3} - \frac{e^{-2a}}{6} \right]$$

$$= I_R = \lim I_R = \mathrm{Re}(\lim I_R)$$

$$\Rightarrow \int_0^{+\infty} \frac{\cos ax}{(x^2+1)(x^2+4)}\, dx = \frac{\pi}{12} e^{-a} \left(2 - e^{-a} \right)$$

a) Probar que

$$\int_0^{2\pi} R(cos\theta, sen\theta) d\theta = -i \int_\gamma R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \frac{dz}{z} \qquad \cancel{ie^{i\theta}}$$

siendo γ la circunferencia unitaria recorrida en sentido antihorario.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(\gamma(\theta) + \frac{1}{\gamma(\theta)} \right)$$

2) $\int_0^{2\pi} \frac{\cos x}{\cos x + \cosh a}\, dx$

$$R(x, y) = \frac{x}{x + \cosh a}$$

$$R\left(\tfrac{1}{2}(z+1/z),\ \tfrac{1}{2i}(z-1/z)\right) = \frac{\frac{1}{2}(z+1/z)}{\frac{1}{2}(z+1/z)+\cosh a}$$

$$= \frac{z^2+1}{z^2+1+2z\cosh a}$$

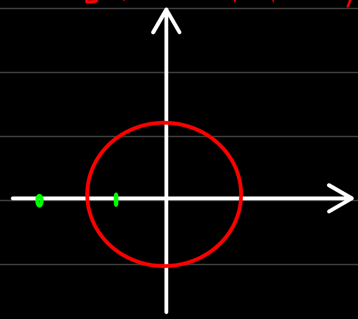
$$z^2 + \cancel{2}z\,\frac{e^a+e^{-a}}{\cancel{2}} + 1 = z^2 - z(-e^a - e^{-a}) + (-e^a)(-e^{-a}) = 0$$

$$(z-\alpha)(z-\beta) = z^2 - z(\alpha+\beta) + \alpha\beta$$

$$\Rightarrow z = -e^a, \quad z = -e^{-a}$$

$a \neq 0$

$a > 0$



$$f(z) = \frac{z^2+1}{(z^2+2\cosh a\, z+1)z}$$

$$\mathrm{Res}(f, -e^{-a}) = \lim_{z\to -e^{-a}} (z+e^{-a})f(z) = \lim_{z\to -e^{-a}} \frac{z^2+1}{(z+e^a)z}$$

$$= \frac{e^{2a}+1}{-(e^a-e^{-a})e^{-a}} = -\frac{e^a+e^{-a}}{e^a-e^{-a}} = -\frac{\cosh a}{\sinh a} = -\coth a$$

Falta $\mathrm{Res}(f, 0)$