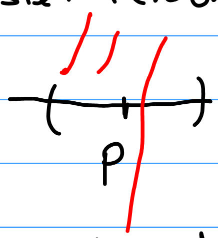
4.2.3.

a) def. de $\lim_{x \to p} f(x) = L$

b) existe un entorno reducido de p donde f está acotada



c) f está acotada en $\mathbb{R}$

d) $\forall x \in \mathbb{R}$, $f(x) = L$.

d) $\Rightarrow$ a, b, c

c) $\Rightarrow$ b)

a) $\Rightarrow$ b)

4.3.1.

a) $$\lim_{x \to 2} \frac{x^2 - x + 2}{x^2 + 4} = \frac{4 - 2 + 2}{4 + 4} = \frac{1}{2}$$

b) $$\lim_{x \to 2} \sqrt{x-1} + 1 = \sqrt{2-1} + 1 = 2$$

g) $$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2} = \frac{1 - 1}{1 - 3 + 2} = \frac{0}{0}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} = \lim_{x \to 1} \frac{x+1}{x-2} = \frac{1+1}{1-2} = -2$$

h) $\lim\limits_{x\to 0} \dfrac{x^4-2x^3}{x^3-x^2} \overset{0/0}{=} \lim\limits_{x\to 0} \dfrac{(x-2)\,x^3}{(x-1)\,x^2} = \lim\limits_{x\to 0} \dfrac{(x-2)\,x}{x-1} = 0$

i) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} \overset{0/0}{=} \lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1} \cdot \dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \lim\limits_{x\to 1} \dfrac{\cancel{x-1}}{(\cancel{x-1})(\sqrt{x}+1)}$

$= \lim\limits_{x\to 1} \dfrac{1}{\sqrt{x}+1} = \dfrac{1}{\sqrt{1}+1} = \dfrac{1}{2}$

j) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} \overset{0/0}{=} \lim \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x} \cdot \dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$= \lim\limits_{x\to 0} \dfrac{1+x-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)} = \lim\limits_{x\to 0} \dfrac{2\cancel{x}}{\cancel{x}\left(\sqrt{1+x}+\sqrt{1-x}\right)} = 1$

n) $\lim\limits_{x\to a} \dfrac{\frac{1}{x}-\frac{1}{a}}{x-a} = \lim\limits_{x\to a} \dfrac{\left(\frac{a-x}{xa}\right)}{x-a} = \lim\limits_{x\to a} \dfrac{-(\cancel{x-a})}{(\cancel{x-a})\,xa}$

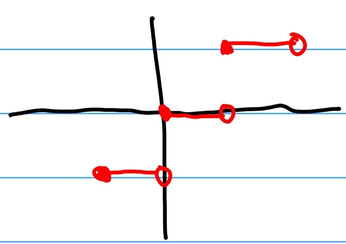
$= \lim\limits_{x\to a} -\dfrac{1}{xa} = -\dfrac{1}{a^2}$

2. a) $\lim\limits_{x\to 0} \lceil x \rceil - \lfloor x \rfloor$ → el entero siguiente → el entero anterior

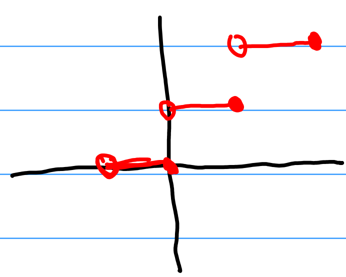
$\lceil x \rceil = \min\{n\in\mathbb{N} \mid x \le n\}$

$\lfloor x \rfloor = \max\{n\in\mathbb{N} \mid n \le x\}$

$\lfloor x \rfloor$



$\lim\limits_{x\to 0} \lfloor x \rfloor$ no existe



$\lceil x \rceil - \lfloor x \rfloor = \begin{cases} 1 & x\in(-1,0) \\ 0 & x=0 \\ 1 & x\in(0,1) \end{cases}$

$\lim\limits_{x\to 0} \lceil x \rceil - \lfloor x \rfloor = 1$

b) $\displaystyle\lim_{x\to 0^+} \frac{\overbrace{\lfloor x \rfloor}^{=0}}{(\underbrace{\lceil x \rceil}_{=1} - x)^3} = \lim_{x\to 0^+} 0 = 0$

$x \in (0,1)$

c) $\displaystyle\lim_{x\to 1} \frac{|(x-1)^3|}{(x-1)^2} \overset{0/0}{=} \lim_{x\to 1} \left| \frac{(x-1)^3}{(x-1)^2} \right| =$

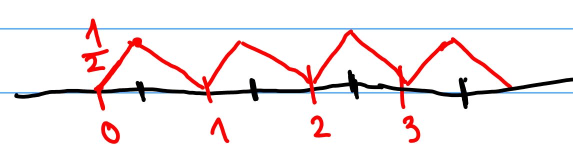
$\displaystyle\lim_{x\to 1} |x-1| = 0$

d) $\displaystyle\lim_{x\to 2} \frac{|x^4-6x^3+12x^2-8x|}{x^5-5x^4+6x^2+4x-8} =$

$$\frac{|\cancel{16}-6\cdot 8+12\cdot 4-\cancel{16}|}{32-5\cdot 16+6\cdot 4+8-8} = \frac{|0|}{56-80} = \frac{0}{-24} = 0$$

e) $\displaystyle\lim_{x\to 0} \frac{|x|^2-|x|}{|x^2-x|} \overset{0/0}{=} \lim_{x\to 0} \frac{|x|(|x|-1)}{|x(x-1)|} =$

$$= \lim_{x\to 0} \frac{\cancel{|x|}(|x|-1)}{\cancel{|x|}\cdot|x-1|} = -1$$



f) $\displaystyle\lim_{x\to 0} \left[\frac{1}{x}\right]$ no existe

si $x = \frac{1}{n}$ $\quad \left[\frac{1}{x}\right] = [n] = 0 \quad n \in \mathbb{Z}$

$x = \frac{1}{n+\frac{1}{2}}$ $\quad \left[\frac{1}{x}\right] = [n+\tfrac{1}{2}] = \frac{1}{2}$

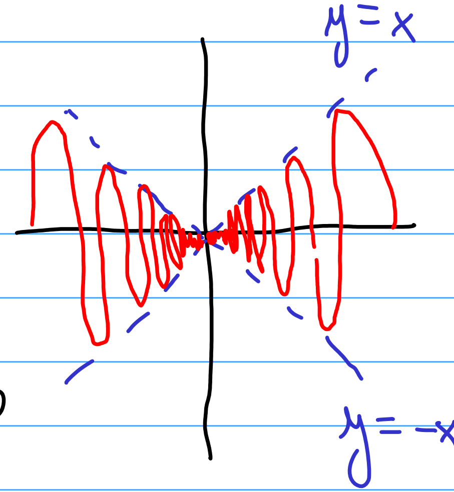
$\forall \delta > 0$, existe $x \in (-\delta, \delta) \setminus \{0\}$ tal que

$f(x) = 0$

$\forall \delta > 0$ existe $y \in (-\delta, \delta) \setminus \{0\}$ tal que

$$f(x) = \frac{1}{2}$$

5. a) $\sin(x) \in [-1, 1] \; \forall x$

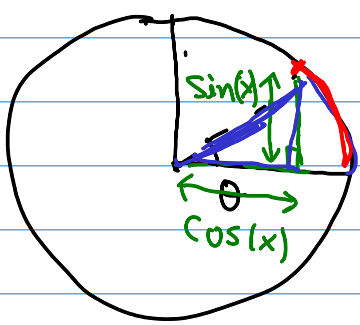


$$\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

$$\left| x \cdot \sin\left(\frac{1}{x}\right) \right| \leq |x| \xrightarrow[x \to 0]{} 0$$

b) $$\lim_{x \to 0^+} \frac{\sin(x)^2}{x} \overset{0/0}{\underset{?}{=}} 0$$

$\rightarrow |\sin(x)| \leq |x|$ $\qquad \left| \frac{\sin(x)^2}{x} \right| \leq \frac{|x|^2}{|x|} = |x| \xrightarrow[x \to 0]{} 0$



x = long. del arco = angulo θ expresado en radianes

c) $\lim\limits_{x\to 0} \frac{\cos(x)-1}{x} = \frac{1-1}{0} = \frac{0}{0}$

$\sin(x)^2 + \cos(x)^2 = 1$

$\cos(x) = \pm\sqrt{1-\sin(x)^2}$ si $x \in (-\delta, \delta)$ tomo el signo $+$

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = \lim_{x\to 0} \frac{\sqrt{1-\sin(x)^2}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1-\sin(x)^2}-1}{x} \cdot \frac{\sqrt{1-\sin^2}+1}{\sqrt{1-\sin^2}+1}$$

$$= \lim_{x\to 0} \frac{1-\sin(x)^2-1}{x\cdot\left(\sqrt{1-\sin(x)^2}+1\right)} = \lim -\underbrace{\frac{\sin(x)^2}{x}}_{\to 0} \cdot \underbrace{\frac{1}{\sqrt{1-\sin(x)^2}+1}}_{\to \frac{1}{2}} = 0$$

6. a) $\lim\limits_{x\to 1} \frac{\log(x)}{x-1} = 1$

$\log(x) = \int_1^x \frac{1}{t}\,dt$

Caso

$x > 1$

$t \in [1, x]$

$\frac{1}{x} \leq \frac{1}{t} \leq 1$

$$\int_1^x \frac{1}{x}\,dt \leq \int_1^x \frac{1}{t}\,dt \leq \int_1^x 1\,dt = x-1$$

$\parallel$

$\frac{1}{x}(x-1)$

Probé $\frac{x-1}{x} \leq \log(x) \leq x-1$

$$\frac{1}{x} \leq \frac{\log(x)}{x-1} \leq 1$$

Tomo $\lim\limits_{x \to 1}$ $\quad 1 \leq \lim\limits_{x \to 1} \frac{\log(x)}{x-1} \leq 1$

$$\Rightarrow \lim_{x \to 1} \frac{\log(x)}{x-1} = 1$$

4.2.2 a) $f(x) = x^2 + \sqrt{x}\ , \quad a = 0$

$$L = \lim_{x \to 0} f(x) = 0 \qquad |x| < \delta$$

$$|f(x) - L| = |x^2 + \sqrt{x}| \leq |x^2| + |\sqrt{x}| \leq \delta^2 + \sqrt{\delta}$$

$$\leq 2\sqrt{\delta} = \varepsilon$$

Va a ser ε

$$\delta = \frac{\varepsilon^2}{4}$$

$\underline{\delta \leq 1} \Rightarrow \delta^2 \leq \sqrt{\delta}$

$$\delta = \min\left(1, \frac{\varepsilon^2}{4}\right)$$

$$\varepsilon = 10^{-5}$$

$$\delta = \frac{10^{-10}}{4}$$