$P=\{a_0,\dots,a_n\}$ $\qquad S^*(f,P)=\sum_{k=0}^{n-1}\overbrace{(a_{k+1}-a_k)}\cdot \sup(f,[a_k,a_{k+1}])$

3.3.1.

$\sup(f,[a_k,a_{k+1}])=\sup\{f(x): x\in[a_k,a_{k+1}]\}$

a) $f:\mathbb{R}\to\mathbb{R},\ f(x)=3x-2$

$P=\{-2,0,1,2\}$

$$\begin{array}{cccc} \vdash & + & + & \dashv \\ -2 & 0 & 1 & 2 \end{array}$$

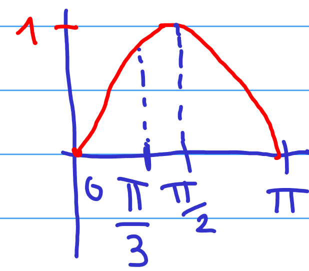
$0-(-2)$

$$S^*(f,P)=2\cdot\sup(f,[-2,0])+1\cdot\sup(f,[0,1])+1\cdot\sup(f,[1,2])$$
$$=2\cdot f(0)+1\cdot f(1)+1\cdot f(2)$$
$$=2\cdot(-2)+1\cdot 1+1\cdot 4=1$$

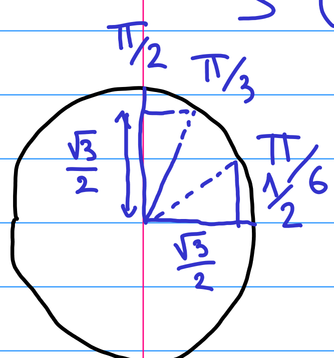
$$S_*(f,P)=2\inf(f,[-2,0])+1\cdot\inf(f,[0,1])+1\cdot\inf(f,[1,2])$$
$$=2\cdot f(-2)+1\cdot f(0)+1\cdot f(1)$$
$$=2(-8)+1\cdot(-2)+1\cdot 1=-17$$

3.3.2 c) $f(x)=\sin(x)$, $P=\{0,\frac{\pi}{3},\frac{\pi}{2},\pi\}$

Si f es creciente en $[a,b]$,
$\inf(f,[a,b])=f(a)$
$\sup(f,[a,b])=f(b)$

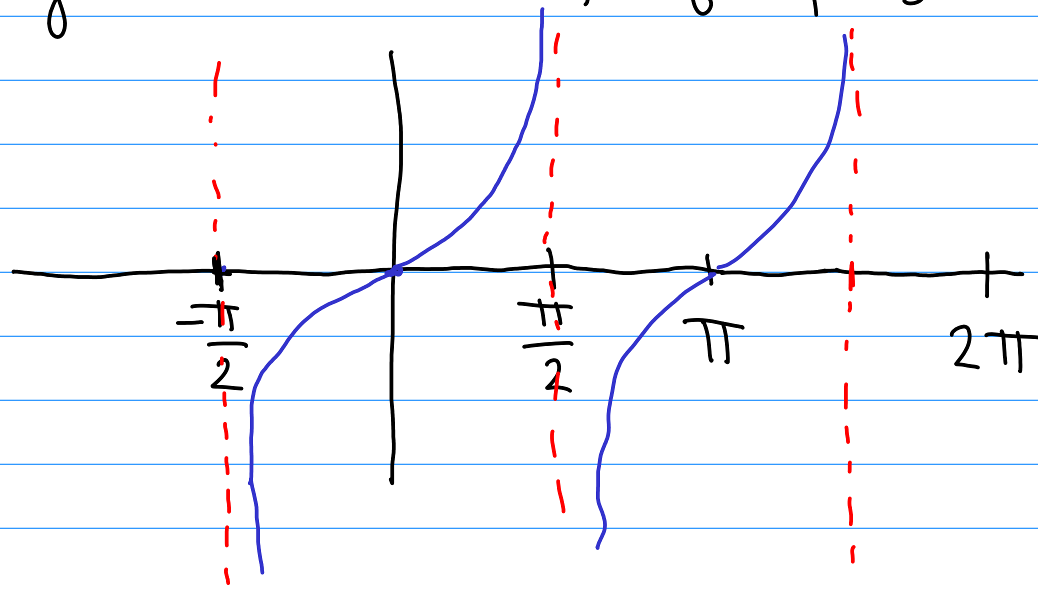


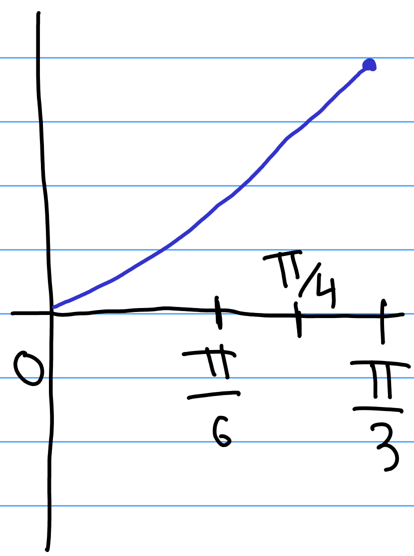
$$S^*(f,P)=\frac{\pi}{3}\cdot\sin\left(\frac{\pi}{3}\right)+\frac{\pi}{6}\cdot 1+\frac{\pi}{2}\cdot 1$$
$$=\frac{\pi}{3}\,\frac{\sqrt{3}}{2}+\frac{\pi}{6}+\frac{\pi}{2}=\pi\left(\frac{\sqrt{3}}{6}+\frac{1}{6}+\frac{1}{2}\right)$$
$$=\pi\,\frac{\sqrt{3}+4}{6}$$



$$S_*(f,P)=\frac{\pi}{3}\cdot 0+\frac{\pi}{6}\cdot\sin\left(\frac{\pi}{3}\right)+\frac{\pi}{2}\cdot 0$$
$$=\frac{\pi}{6}\cdot\frac{\sqrt{3}}{2}=\frac{\pi\sqrt{3}}{12}$$

e) $f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$, $P = \{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\}$

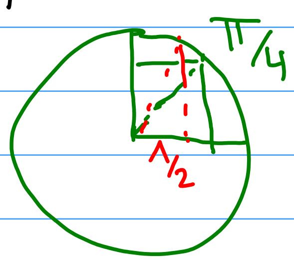




$$S^*(f,P) = \frac{\pi}{6} \cdot \tan\left(\frac{\pi}{6}\right) + \frac{\pi}{12} \tan\left(\frac{\pi}{4}\right) + \frac{\pi}{12} \tan\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6} \cdot \frac{1}{\sqrt{3}} + \frac{\pi}{12} + \frac{\pi}{12} \cdot \sqrt{3}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = 1$$

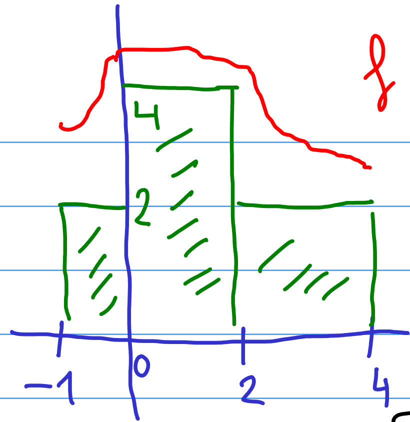


$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

3.2.2 $f : [-1, 4] \to \mathbb{R}$ integrable

- $f(x) \geq 2 \quad \forall x \in [-1, 0] \cup [2, 4]$
- $f(x) \geq 4 \quad \forall x \in [0, 2]$



$$\int_{-1}^{4} f(x)\,dx \geq \text{área verde} = 14$$

Propiedad $\forall$ partición P de $[-1,4]$

$$S_*(f,P) \leq \int_{-1}^{4} f(x)\,dx \leq S^*(f,P)$$

Elijo $P = \{-1, 0, 2, 4\}$

$$\int_{-1}^{4} f(x)\,dx \geq S_*(f,P) = 1\cdot \inf(f,[-1,0]) + 2\cdot \inf(f,[0,2]) + 2\cdot \inf(f,[2,4]) \quad (*)$$

$\forall x \in [-1,0],\ f(x) \geq 2 \Rightarrow \inf(f,[-1,0]) \geq 2$

$\inf(f,[0,2]) \geq 4$

$\inf(f,[2,4]) \geq 2$

$$(*) \geq 1\cdot 2 + 2\cdot 4 + 2\cdot 2 = 14$$

3.2.3. $f:\mathbb{R}\to\mathbb{R}$ monótona creciente e integrable

Probar $\displaystyle\sum_{k=0}^{n-1} f(k) \leq \int_0^n f(x)\,dt \leq \sum_{k=1}^{n} f(k)$ $\quad$ $n\in\mathbb{N}$

$$\sum_{k=1}^{n} f(k) = \sum_{k=0}^{n-1} f(k+1)$$

$P_n = \{0, 1, 2, \ldots, n-1, n\}$

$$S^*(f,P_n) = \sum_{k=0}^{n-1} 1\cdot \sup(f,[k,k+1]) = \sum_{k=0}^{n-1} f(k+1)$$

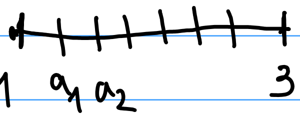
$$S_*(f,P_n) = \sum_{k=0}^{n-1} 1 \cdot \inf(f,[k,k+1]) = \sum_{k=0}^{n-1} f(k)$$

$$\Rightarrow \quad \underset{\sum_{k=0}^{n-1} f(k)}{\underset{\parallel}{S_*(f,P_n)}} \leq \int_0^n f(t)\,dt \leq \underset{\sum_{k=1}^{n} f(k)}{\underset{\parallel}{S^*(f,P_n)}}$$

3.3.5. Calcular $\int_1^3 x\,dx$ con sumas sup. e inf. para particionos equiespaciados

Partición equiespaciada $P_n = \{a_0, \dots, a_n\}$

$\overset{2/n}{\leftrightarrow}$ n intervalos de mismo tamaño $\rightarrow \frac{2}{n}$



$a_0 = 1$

$a_1 = 1 + \frac{2}{n}$

$a_2 = 1 + 2 \cdot \frac{2}{n}$

$\rightarrow a_k = 1 + k \cdot \frac{2}{n}$, $a_n = 1 + 2 = 3$ ✓

$\forall n \in \mathbb{N}$, $\quad S_*(x,P_n) \leq \int_1^3 x\,dx \leq S^*(x,P_n)$

Cuenta: $S_*^{*}(x,P_n) = \sum_{k=0}^{n-1} (a_{k+1} - a_k) \cdot \overset{\sup}{\inf}(x,[a_k,a_{k+1}])$

$= \sum_{k=0}^{n-1} (a_{k+1} - a_k) \cdot a_{k+1} = \sum_{k=0}^{n-1} \frac{2}{n} \cdot a_{k+1} = \sum_{k=0}^{n-1} \frac{2}{n}\left(1 + \underset{k+1}{k}\frac{2}{n}\right)$

$= \frac{2}{n} \cdot \sum_{k=0}^{n-1} \left(1 + \overset{k+1}{k}\frac{2}{n}\right) = \frac{2}{n}\left(\sum_{k=0}^{n-1} 1 + \sum_{k=0}^{n-1} \overset{k+1}{k} \cdot \frac{2}{n}\right)$

$$= \frac{2}{n}\left(n + \frac{2}{n}\cdot \sum_{k=0}^{n-1} k^{+1}\right)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n-1} k = \frac{(n-1)\cdot n}{2}$$

$$= \frac{2}{n}\left(n + \frac{\cancel{2}}{\cancel{n}}\,\frac{\overset{n+1}{(n-1)}\cdot \cancel{n}}{\cancel{2}}\right) = \frac{2}{n}\left(2n \overset{+}{-} 1\right) = 4 \overset{+}{-} \frac{2}{n}$$

$$S^*(f, P_n) = 4 + \frac{2}{n}$$

$$\forall n \in \mathbb{N}, \quad 4 - \frac{2}{n} \leq \int_1^3 x\,dx \leq 4 + \frac{2}{n}$$

Pasando al límite $n \to +\infty$, obtengo

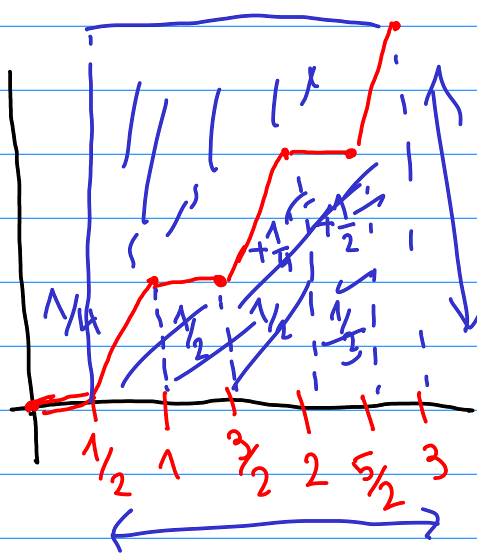
$$4 \leq \int_1^3 x\,dx \leq 4$$

o sea $\int_1^3 x\,dx = 4.$

$x \in [n, n+\frac{1}{2})$

o $[n+\frac{1}{2}, n+1)$

3.1.6.c)



$$\int_0^x f_3(t)\,dt =$$

$$\int_0^n f_3(t)\,dt + \int_n^x f_3(t)\,dt$$

$$\frac{\left(n - \frac{1}{2}\right)\cdot n}{2}$$

$n=1 \to \frac{1}{4}$

$n=2 \to \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$

$n=3 \to 3\cdot\frac{1}{4} + 4\cdot\frac{1}{2}$