3. Determinar y bosquejar los $z \in \mathbb{C}$ tales que

a) $|z - 1 + i| \leq 4$

b) $|z+1| \leq 4 - |z-1|$. $z=0$: $|1| = 1 \leq 4-1$ ✓

$|z|^2 = z \cdot \bar{z}$

$|z+1| = 4 - |z-1| \Leftrightarrow$

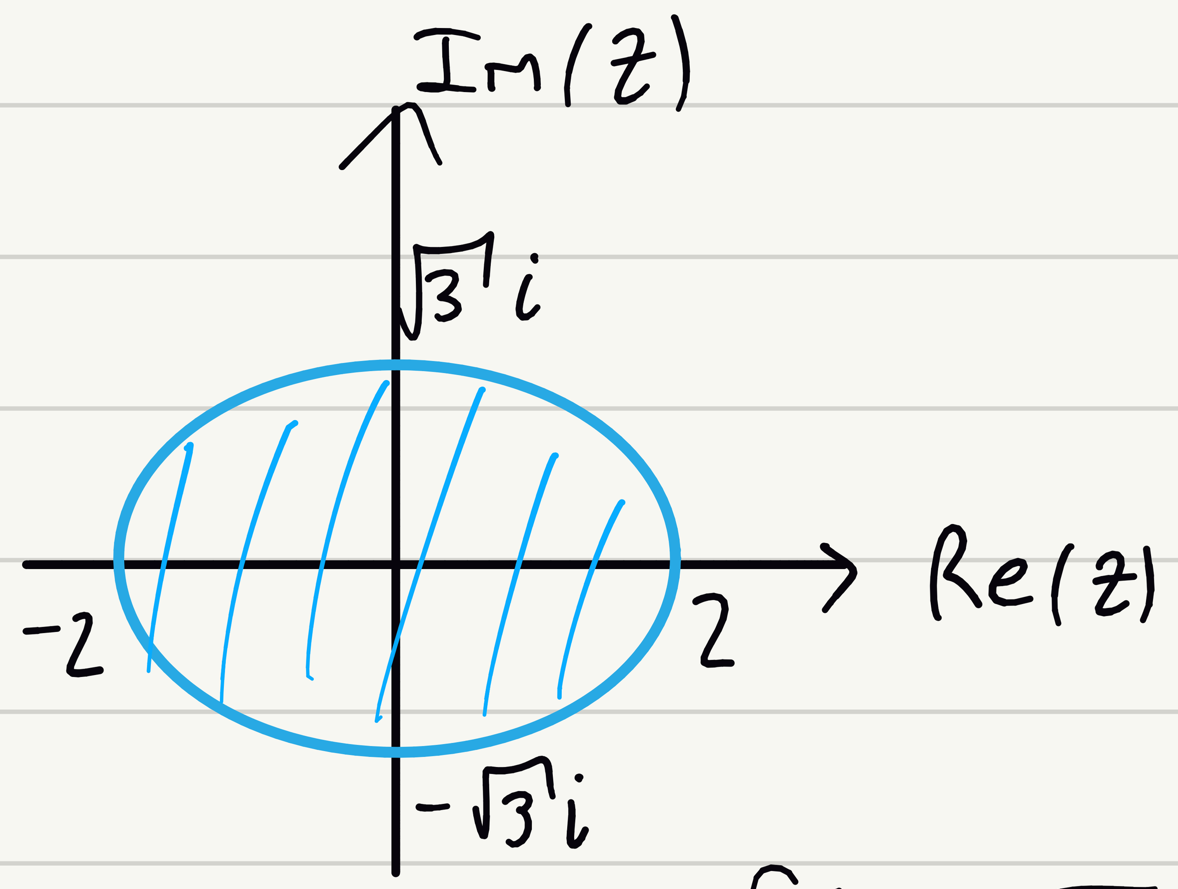
$(z+1)\overline{(z+1)} = 16 - 8|z-1| + (z-1)\overline{(z-1)}$

$\Leftrightarrow \cancel{z\bar{z}} + z + \bar{z} + \cancel{1} = 16 - 8|z-1| + \cancel{z\bar{z}} - z - \bar{z} + \cancel{1}$

$\Leftrightarrow 4\operatorname{Re}(z) = 16 - 8|z-1| \Leftrightarrow 16 - 4x = 8|z-1| \Leftrightarrow 4 - x = 2|z-1|$

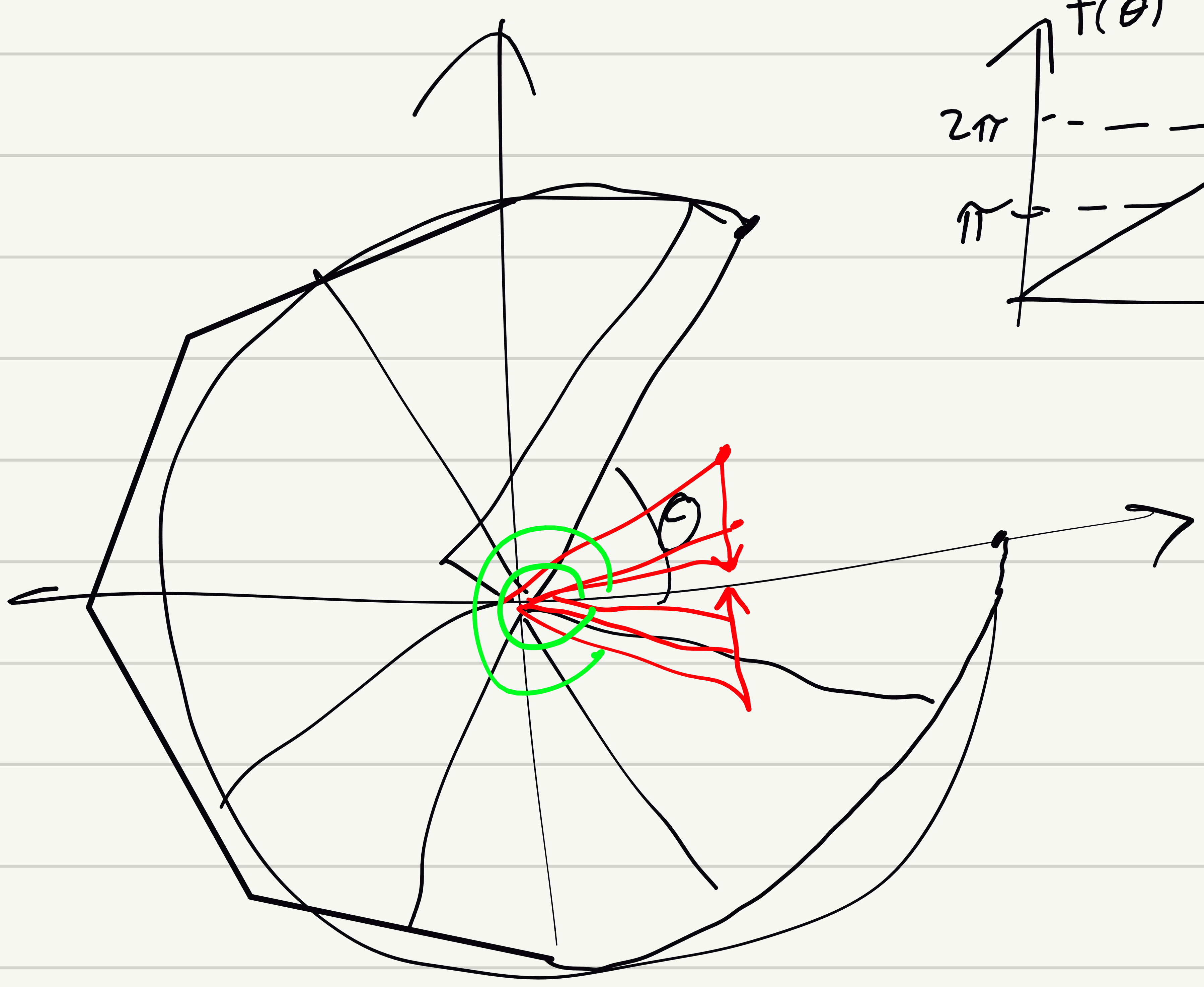
$\Leftrightarrow 16 - \cancel{8x} + x^2 = 4|x-1+iy|^2 = 4(x-1)^2 + 4y^2 = 4x^2 - \cancel{8x} + 4 + 4y^2 \Leftrightarrow$

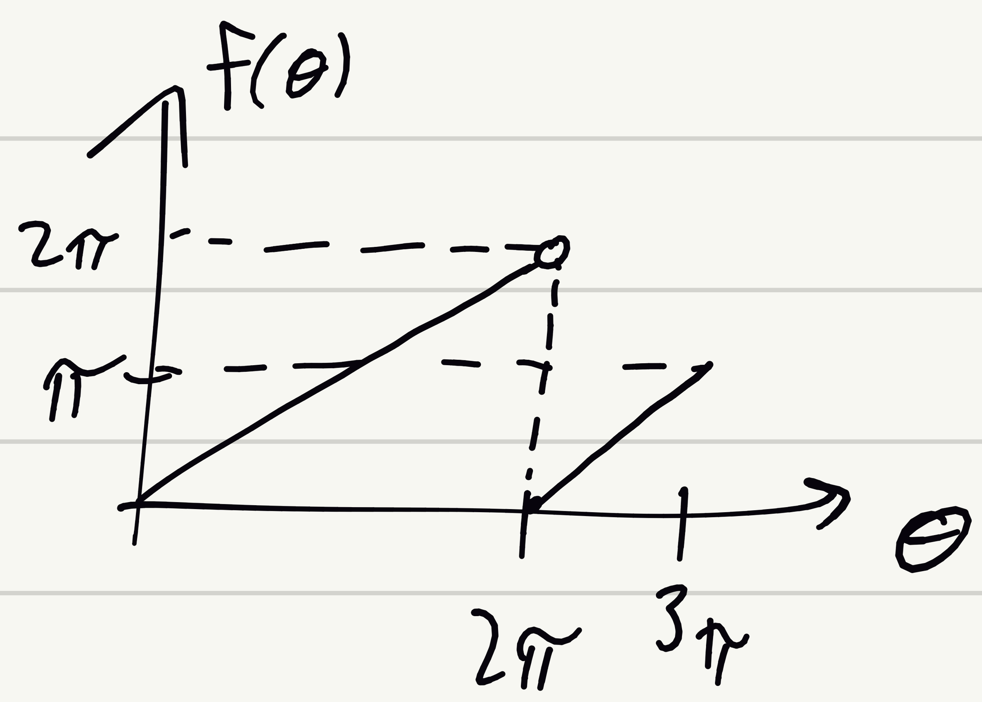
$3x^2 + 4y^2 = 12 \Leftrightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1 \Leftrightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1$



$\left\{ z = x + iy : \left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{3}}\right)^2 = 1 \right\}$

$f(\theta) = \operatorname{Im}(\log(e^{i\theta}))$, $\theta \in [0, 3\pi]$





$$\log(z^n) = \ln|z^n| + i\,\mathrm{Arg}(z^n)$$
$$= \ln(|z|^n) + i\,\mathrm{Arg}(z^n)$$
$$= n\ln|z| + i(n\,\mathrm{Arg}(z) \bmod 2\pi)$$

$$\mathrm{Arg}(-1) = \pi$$
$$3\,\mathrm{Arg}(-1) = 3\pi$$
$$3\,\mathrm{Arg}(-1) \bmod 2\pi = \pi$$

$$\mathrm{Img}(\log) \subseteq \{z \in \mathbb{C} : \mathrm{Im}(z) \in [0, 2\pi)\}$$

14. Dados tres números complejos a, b y c de módulo 1 y tales que $a+b+c=0$ muestre que los mismos (como puntos del plano) forman un triángulo equilátero.

$|a| = 1 \Rightarrow a \neq 0$ $\qquad |b/a| = |b|/|a| = 1 = |c|/|a| = |c/a|$

$a+b+c=0 \Leftrightarrow 1 + b/a + c/a = 0 \Leftrightarrow 1+z+w=0,\ |z|=|w|=1$

$1+z=-w \Rightarrow |1+z| = |-w| = |w| = 1$

$z = e^{i\theta} \Rightarrow 1+z = (\cos\theta+1) + i\sin\theta \Rightarrow (\cos\theta+1)^2 + \sin\theta^2 = 1$

$\cos^2\theta + 2\cos\theta + \cancel{1} + \sin^2\theta = \cancel{1}$

$1 + 2\cos\theta = 0 \Rightarrow \cos\theta = -1/2$

$$z^n - 1 = (z-1)(z-z_2)(z-z_3)\cdots(z-z_n) \qquad z_k = e^{i\frac{2\pi}{n}k}$$

$$\frac{z^n-1}{z-1} = (z-z_2)\cdots(z-z_n) \Big\rangle f(z)$$

$$= 1 + z + z^2 + \cdots + z^{n-1}$$

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$$

$$|f(z)| = \prod_{k=2}^{n} |z - z_k| \Rightarrow \underset{\underset{|n| = n}{\|}}{|f(1)|} = \prod_{k=2}^{n} |1 - z_k| = \prod_{k=2}^{n} \lambda_k$$

$$P_n = n \cdot \ell_n \qquad \left|1 - e^{i\frac{2\pi}{n}}\right| = \left|\cos\left(\tfrac{2\pi}{n}\right) - 1 + i\sin\left(\tfrac{2\pi}{n}\right)\right|$$

$$\| \quad \sqrt{\cos^2 + \sin^2 - 2\cos + 1} = \sqrt{2 - 2\cos}$$

$$\lim_n n\sqrt{2 - 2\cos\left(\tfrac{2\pi}{n}\right)}$$

$$= \sqrt{\lim_n n^2\left(\cancel{2} \cancel{+2}\left(\cancel{1} + \frac{1}{\cancel{2}}\left(\frac{2\pi}{n}\right)^2 + r_2\left(\tfrac{2\pi}{2}\right)\right)\right)}$$

$$= \sqrt{\lim_n \cancel{n^2}\left(\frac{2\pi}{\cancel{n}}\right)^2 + n^2\, r_2\left(\frac{2\pi}{n}\right)}$$

$$= \sqrt{\lim_n 4\pi^2 + \cancel{n^2}\frac{(2\pi)^2}{\cancel{n^2}}\, \cancel{\frac{r_2(2\pi/n)}{(2\pi)^2/n^2}} \to 0} = 2\pi$$