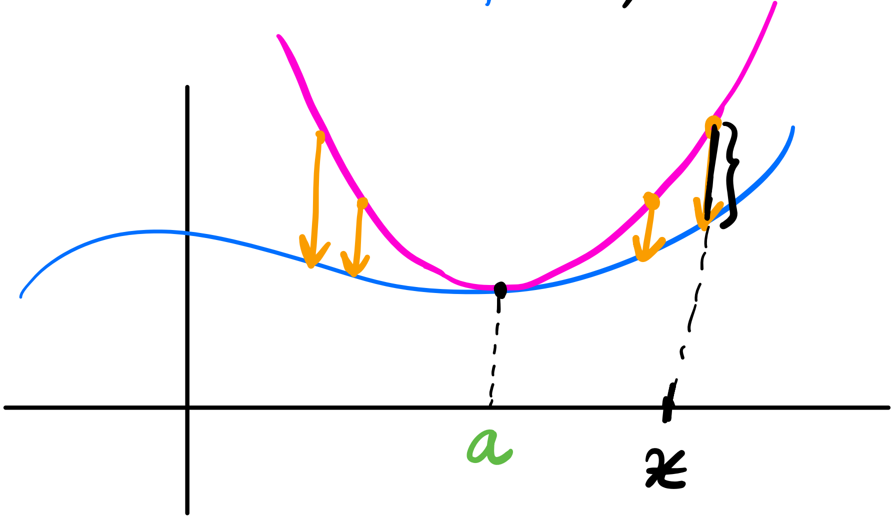
$$f(x) = P_n(f,a)(x) + R_n(f,a)(x)$$



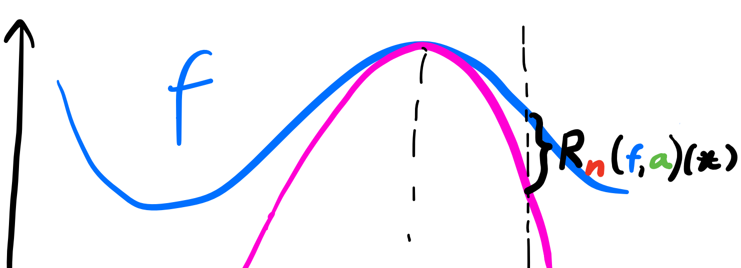
Fórmula del Resto de Lagrange

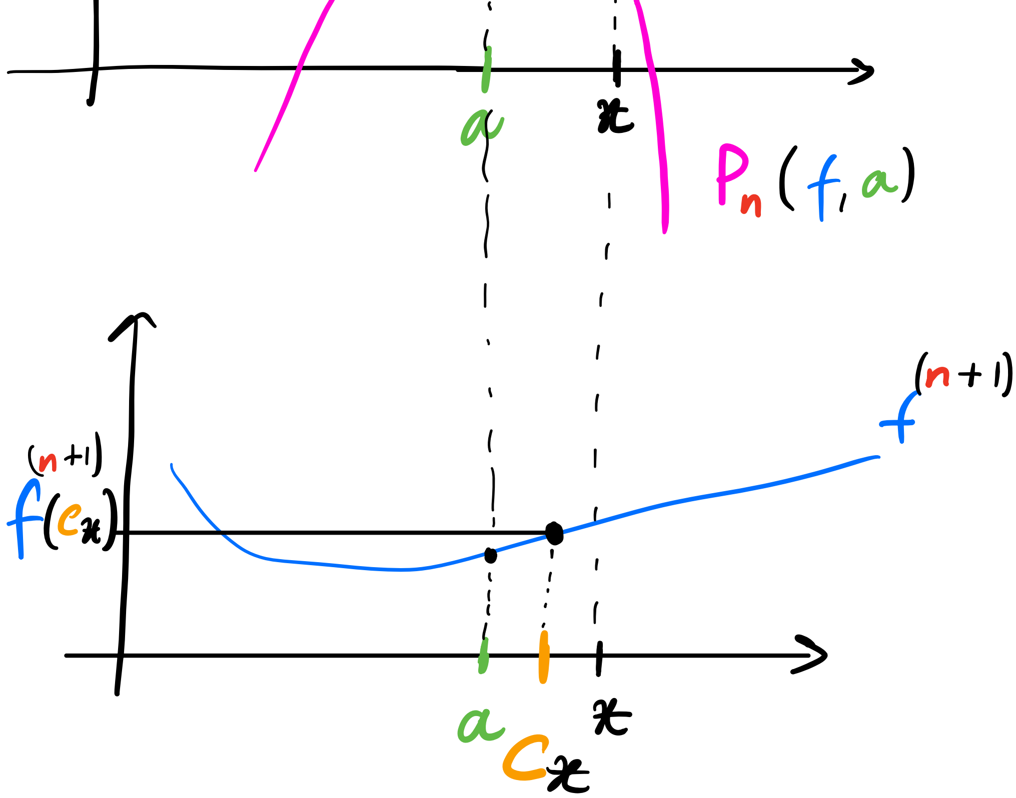
Sea $f : I \longrightarrow \mathbb{R}$ $(n+1)$-veces derivable en I, y $a \in I$.

Entonces, $R_n(f,a)(x) = \dfrac{f^{(n+1)}(c_x)(x-a)^{n+1}}{(n+1)!}$

con c_x está entre a y x

$$\left(\begin{array}{l} c_x \in (a,x) \text{ si } a < x \\ c_x \in (x,a) \text{ si } x < a \end{array} \right)$$





Ejemplo

$$f(x) = \operatorname{sen}(x)$$

$$P_n(f,0) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$P_{2n+1}(f,0) = \sum_{i=0}^{i=n} \frac{(-1)^i}{(2i+1)!} x^{2i+1}$$

$n=0 \quad \frac{(-1)^0}{(2.0+1)!} x^{2.0+1} = x$

$n=1 \quad x + \frac{(-1)^1}{(2.1+1)!} x^{2.1+1} = x - \frac{x^3}{3!}$

$$R_{2n+1}(f,0)(x) = \frac{f^{(2n+2)}(c_x)\,(x-0)^{2n+2}}{(2n+2)!} \qquad (x>0)$$

donde $c_x \in (0, x)$

Como $f = \operatorname{sen}(x)$

$$f^{(k)} = \begin{cases} \operatorname{sen} \\ \cos \\ -\operatorname{sen} \\ -\cos \end{cases} \quad \text{dependiendo de } k$$

En particular, podemos afirmar que

$$\left| f^{(2n+2)}(c_x) \right| \leq 1$$

Entonces

$$\left| R_{2n+1}(f,0)(x) \right| = \left| \frac{f^{(2n+2)}(c_x)\,(x-0)^{2n+2}}{(2n+2)!} \right| \overset{(*)}{\leq} \frac{x^{2n+2}}{(2n+2)!}$$

Porque $\left| f^{(2n+2)}(c_x) \right| \leq 1$ entonces, multiplicando por $\dfrac{x^{2n+2}}{(2n+2)!}$ a ambos lados obtenemos $(*)$

Calculemos una aproximación de $\operatorname{sen}(2)$ con un error menor a $\dfrac{1}{1000}$

Queremos encontrar $2n+1 \in \mathbb{N}$ /

$$\Big|\underbrace{\operatorname{sen}(2) - P_{2n+1}(\operatorname{sen},0)(2)}_{R_{2n+1}(f,0)(2)}\Big| < \frac{1}{1000}$$

Por ⊛ sabemos que $\left|R_{2n+1}(f,0)(2)\right| \leq \dfrac{2^{2n+2}}{(2n+2)!}$

Si $n=4$, tenemos que

$$\frac{2^{2.4+2}}{(2.4+2)!} = \frac{2^{10}}{10!} = \frac{4}{14175} < \frac{1}{1000}$$

Entonces $\Big|R_{\underbrace{2.4+1}_{9}}(f,0)(2)\Big| < \dfrac{1}{1000}$

Calculando, obtenemos

$$P_9(\operatorname{sen},0)(2) = \frac{2578}{2835} = 0{,}909347\ldots\ldots$$

Entonces, sabemos que

$$\left|\operatorname{sen}(2) - \frac{2578}{2835}\right| = \left|R_9(\operatorname{sen},0)(2)\right| \leq \frac{1}{1000}$$