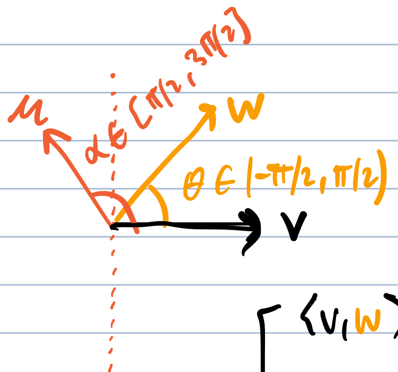
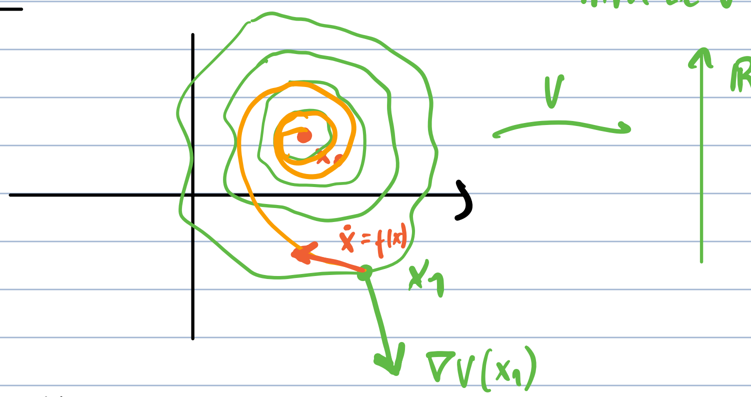
Recordamos: Si $\dot{x}=f(x)$, x_0 pto crítico, decimos que $V: U_{x_0} \to \mathbb{R}$ es función de Lyapunov si x_0 es mínimo estricto de V y <u>V decrece en las órbitas solución</u>

$\hookrightarrow \frac{d}{dt}\big(V(x(t))\big) \le 0$ sii $\boxed{\nabla V(x(t)) \cdot \dot{x}(t) \le 0}$

<u>interpretación geométrica:</u>

En verde conj de nivel de V.





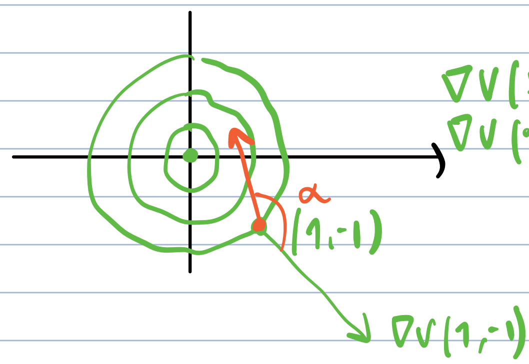
$\langle v, w \rangle = |v||w| \cos\theta \ge 0$

$\langle v, u \rangle = |v||u| \cos\alpha \le 0$

<u>Ejemplo</u>: $V(x,y) = x^2 + y^2$

- $(\dot{x}, \dot{y}) = (-x, -2y)$

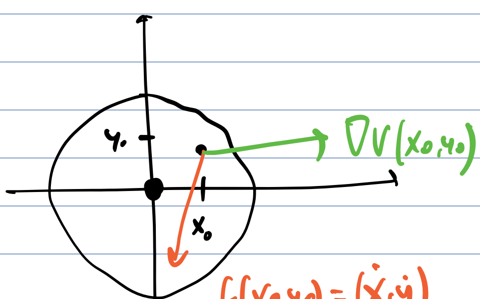
$f(1,-1) = (-1, 2)$

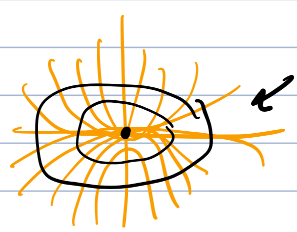


$\nabla V(x,y) = (2x, 2y)$
$\nabla V(1,-1) = (2,-2)$

En este caso $\nabla V(1,-1) \cdot (-1,2) = (2,-2) \cdot (-1,2) = -2-4 = -6$

$= |(2,-2)| \cdot |(-1,1)| \cdot \cos(\alpha) = -4.$





$f(x_0,y_0) = (\dot{x}, \dot{y})$

donde $(x(t), y(t))$ soluc a $\begin{cases} (\dot{x},\dot{y}) = f(x,y) \\ (x(t_0), y(t_0)) = (x_0, y_0) \end{cases}$

8) Vamos a buscar funciones de Lyapunov de la forma $V(x,y) = ax^2 + by^2$

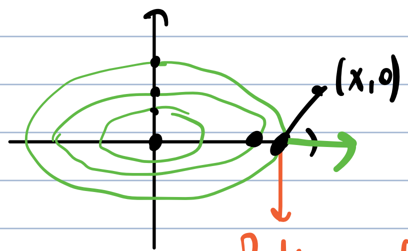
a) $\begin{cases} \dot{x} = 3xy \\ \dot{y} = -x^2 - y^3 \end{cases}$

· Estudiamos el crítico $(0,0)$ (mínimo de V).

· $\nabla V(x,y) \cdot (\dot{x}, \dot{y}) \leq 0$ sii

$(2ax, 2by) \cdot (3xy, -x^2 - y^3) = 6ax^2y - 2bx^2y - 2by^4$

$= x^2y(\cancel{6a - 2b}) - y^4(2b) \leq 0$



(no estricta, obs que $\nabla V(x,0) \cdot (0, -x^2) = 0$)

• Podemos tomar $b = 1$, $a = \frac{1}{3}$.

(Sirven los pares (a,b) con $b > 0$ y $a = b/3$)

b) $\begin{cases} \dot{x} = -x^3 + xy^2 \\ \dot{y} = -2x^2y - y^3 \end{cases}$

$V(x,y) = ax^2 + by^2$.

$\nabla V(x,y) \cdot (f(x,y)) = (2ax, 2by) \cdot (-x^3 + xy^2, -2x^2y - y^3)$

$= -2ax^4 + 2ax^2y^2 - 4bx^2y^2 - 2by^4$

$= x^4\underbrace{(-2a)} + x^2y^2\underbrace{(2a - 4b)} + y^4\underbrace{(-2b)}$

$a > 0$, $b > 0$ tq $2a - 4b < 0$, $a < 2b$ $\longrightarrow$ < 0.

Eg: $a = 1$, $b = 1$

(Función de Lyapunov estricta)

• <u>Linealización en un pto crítico</u>:

· Consideramos $\dot{x} = f(x)$ con $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ de clase C^1

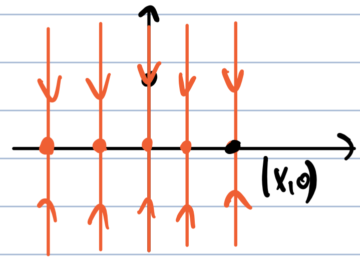
Asumimos que x_0 es un pto crítico, consideramos el sistema lineal

$\dot{x} = Ax$ donde $A = [df_{x_0}] = Jac(f, x_0)$

• Entonces, • Si A tiene λ vap con $Re(\lambda) > 0 \Rightarrow x_0$ es inestable.

• Si $Re(\lambda) < 0$ para todo λ vap $\Rightarrow x_0$ es asint. estable.

6) a) • $\begin{cases} \dot{x} = 0 \\ \dot{y} = -y \end{cases}$



• $f(x,y) = (0, -y)$

• $Jf_{(x,y)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

• $Jf_{(x,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

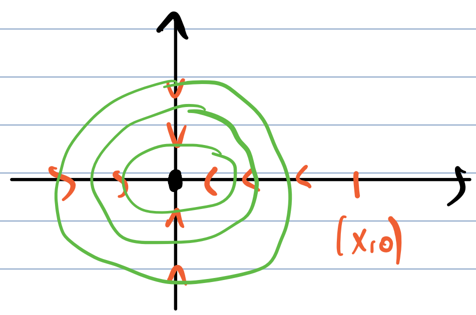
Observar que $(\dot{x}, \dot{y}) = Jf_{(x,0)}(x,y)$

$$(\dot{x}, \dot{y}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (0, -y)$$

c) • $\begin{cases} \dot{x} = -x^3 \\ \dot{y} = -y \end{cases}$

$f(x,y) = (-x^3, -y)$

$Jf_{(x,y)} = \begin{pmatrix} -3x^2 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow Jf_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$



• $V(x,y) = ax^2 + by^2$

$\nabla V(x,y) \cdot (f(x,y)) = (2ax, 2by) \cdot (-x^3, -y)$

$= -2ax^4 - 2by^2 < 0$

Si $(x,y) \neq (0,0)$, $a > 0$, $b > 0$