Fórmula de integración por partes

$$\int f'(x).g(x)\,dx \overset{(*)}{=} f(x).g(x) - \int f(x).g'(x)\,dx$$

Versión definida de integración por partes

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx$$

Deducción de la versión definida de la fórmula:

Por Barrow $\displaystyle\int_a^b f'(x)g(x)\,dx = F(x)\Big|_a^b$ donde

F es una primitiva de $f'(x)g(x)$

Por $(*)$ podemos tomar como

$F(x) = f(x).g(x) - \varphi$ donde $\varphi = \int f(x)g'(x)\,dx$

Entonces

$$\boxed{\int_a^b f'(x)g(x)\,dx} = \overbrace{\left(f(x).g(x) - \varphi\right)}^{F}\Big|_a^b =$$

$$= \left(f(b)g(b) - \varphi(b)\right) - \left(f(a)g(a) - \varphi(a)\right) =$$

$$= \left(f(b)g(b) - f(a)g(a)\right) - \left(\varphi(b) - \varphi(a)\right) =$$

$$= \left(f(b)g(b) - f(a)g(a)\right) - \int_a^b f(x)g'(x)\,dx =$$

Barrow

$$= \boxed{f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx}$$

$$\int_0^{\pi/2} \underbrace{x}_{v} \cdot \underbrace{\text{sen}(x)}_{u'} = \underbrace{-\cos(x)}_{u} \cdot \underbrace{x}_{v}\Big|_0^{\pi/2} - \int_0^{\pi/2} \underbrace{-\cos(x)}_{u} \cdot \underbrace{1}_{v'}\,dx =$$

$$\boxed{\begin{array}{l} v = x \Rightarrow v' = 1 \\ u' = \text{sen}(x) \Rightarrow u = -\cos(x) \end{array}}$$

$$= -\cos(x)\,x\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos(x)\,dx =$$

$$= \left(-\overbrace{\cos(\pi/2)}^{0}\,\pi/2 - \left(-\cos(0)\cdot 0\right)\right) + \overbrace{\int_0^{\pi/2} \cos(x)\,dx}^{1 \,\overset{\circledast}{=}} = 1$$

$$\boxed{\int_0^{\pi/2} \cos(x)\,dx = \text{sen}(x)\Big|_0^{\pi/2} = \text{sen}(\pi/2) - \text{sen}(0) = 1} \circledast$$

Barrow

Ejemplos

$$\int uv' = uv - \int u'v$$

$$\int \text{sen}^2(x)\,dx = \int \underbrace{\text{sen}(x)}_{u} \cdot \underbrace{\text{sen}(x)}_{v'}\,dx \overset{\circledast}{=}$$

$u = \text{sen}(x) \Rightarrow u' = \cos(x)$

$v' = \text{sen}(x) \Rightarrow v = -\cos(x)$

$$= \underbrace{\text{sen}(x)}_{=U} \cdot \underbrace{(-\cos(x))}_{=V} + \int \underbrace{\cos(x)}_{=U'} \cdot \underbrace{(+\cos(x))}_{=V} \, dx =$$

$$= -\text{sen}(x)\cos(x) + \int 1 - \text{sen}^2(x)\, dx = -\text{sen}(x)\cos(x) + x - \int \text{sen}^2(x)\, dx$$

$\boxed{\text{sen}^2 + \cos^2 = 1}$

$\int 1 - \text{sen}^2(x)\, dx = \int 1 dx - \int \text{sen}^2(x)\, dx =$
$= x - \int \text{sen}^2(x)\, dx$

$$\int \text{sen}^2(x)\, dx = \left[-\text{sen}(x)\cos(x) + x\right] - \int \text{sen}^2(x)\, dx$$

$$\Rightarrow \int \text{sen}^2(x)\, dx = \frac{\left[-\text{sen}(x)\cos(x) + x\right]}{2}$$

Verificación:

$$\left(\frac{(-\text{sen}(x))(\cos(x)) + x}{2}\right)' = \frac{1}{2}\left((-\cos(x)) \cdot \cos(x) + (-\text{sen}(x))(-\text{sen}(x)) + 1\right)$$

$$= \frac{1}{2}\left(\underbrace{1 - \cos^2(x)}_{=\text{sen}^2(x)} + \text{sen}^2(x)\right) = \frac{1}{\cancel{2}} \cdot \cancel{2}\, \text{sen}^2(x) = \text{sen}^2(x)$$

MAGIA: HACER APARECER UN 1

$$\int \log(x)\, dx = \int \underbrace{1}_{=V'} \cdot \underbrace{\log(x)}_{=U}\, dx = x \log(x) - \int \frac{1}{x} \cdot x\, dx =$$

$$\boxed{\begin{array}{l} v' = 1 \Rightarrow v = x \\ u = \log(x) \Rightarrow u' = \frac{1}{x} \end{array}}$$

$$= x\log(x) - \int 1\,dx = x\log(x) - x =$$

$$= x(\log(x) - 1)$$

INTEGRACIÓN POR SUSTITUCIÓN O CAMBIO DE VARIABLE

f continua, g derivable con derivada continua

$$\int f(g(x)).g'(x)\,dx = F(g(x))$$

donde $F' = f$

Ejemplo:

$$\int e^{\operatorname{sen}(x)} . \cos(x)\,dx$$

$e^{\operatorname{sen}(x)} . \cos(x) = f(g(x))g'(x)$ donde $\left[\begin{array}{l} f = e^{x} \\ g = \operatorname{sen}(x) \Rightarrow g' = \cos(x) \end{array}\right.$

$$F(x) = \int \underbrace{e^{x}}_{= f} dx = e^{x}$$

Entonces, por la fórmula de sustitución:

$$\int e^{\operatorname{sen}(x)} \cdot \cos(x)\, dx = F(g(x)) = e^{\operatorname{sen}(x)}$$

$$\int f(g(x))g'(x) = F(g(x)) \quad \text{donde } F' = f$$

OTRO Ejemplo:

$$\int (\operatorname{sen}(x))^2 \cdot \cos(x)\, dx$$

$$(\operatorname{sen}(x))^2 \cdot \cos(x) = f(g(x)) \cdot g'(x) \quad \text{donde} \quad f = x^2$$

$$g = \operatorname{sen}(x) \Rightarrow g' = \cos(x)$$

$$F(x) = \int x^2 dx = \frac{x^3}{3}$$

Entonces, por la fórmula de sustitución:

$$\int (\operatorname{sen}(x))^2 \cos(x)\, dx = \boxed{\frac{(\operatorname{sen}(x))^3}{3}}$$

Nemotecnia:

$$\int (\underbrace{\operatorname{sen}(x)}_{U})^2 \cdot \cos(x)\, dx = \int U^2\, dU = \frac{U^3}{3} = \boxed{\frac{(\operatorname{sen}(x))^3}{3}}$$

$U = \operatorname{sen}(x)$

$dU = \cos(x)\, dx$

deshacer el cambio de variable

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2\int \overbrace{e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}^{f(g(x))g'(x)}\, dx = 2\int \overbrace{e^{U}}^{f} dU =$$

$U = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}dx$

$$= 2\overbrace{e^{u}}^{F} = \boxed{2 . e^{\sqrt{x}}}$$

deshacemos el cambio de variable

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int 2x . \sqrt{1+x^2}\,dx \underset{\substack{u = 1+x^2 \\ du = 2x\,dx}}{=} \frac{1}{2}\int \sqrt{u}\,du =$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{2}\,\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = \frac{1}{3}u^{\frac{3}{2}} \underset{D.C.V}{=} \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}$$

$$\int u^{\alpha} = \frac{u^{\alpha+1}}{\alpha+1}$$