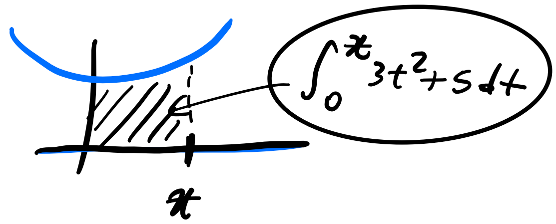
DERIVADAS DE FUNCIONES DADAS POR EXPRESIONES INTEGRALES

$$F(x) = \int_0^x 3t^2+5 \, dt$$



Por el teorema fundamental del cálculo

$$F'(x) = 3x^2+5$$

Qué pasa si queremos derivar

$$F(x) = \int_x^0 3t^2+5 \, dt \quad ?$$

$$F'(x) = \left(\int_x^0 3t^2+5\,dt\right)' = \left(-\int_0^x 3t^2+5\,dt\right)' \overset{TFC}{=} -(3x^2+5)$$

Y si queremos derivar

$$F(x) = \int_0^{x^2} 3t^2+5\,dt$$

Para poder derivar $F(x)$ vamos a escribirla como composición

$$\int_0^{x^2} 3t^2+5\,dt = G(x^2) \quad \text{donde} \quad G(x) = \int_0^x 3t^2+5\,dt$$

$$\overset{TFC}{\hookrightarrow} G'(x) = 3x^2+5$$

$$F'(x) = \left(\int_0^{x^2} 3t^2+5\,dt\right)' = \left(G(x^2)\right)' \underset{1}{=} G'(x^2) \cdot 2x =$$

Regla de la Cadena

$$= \left(3\left(x^2\right)^2 + 5\right) . 2x = \left(3x^4 + 5\right) 2x = 6x^5 + 10x$$

$$F(x) = \int_{\cos(x)}^{e^x} 3t^2 + 5\, dt = \int_{\cos(x)}^{0} 3t^2 + 5\, dt + \int_0^{e^x} 3t^2 + 5\, dt =$$

Aditividad Respecto del Intervalo

$$= -\int_0^{\cos(x)} 3t^2 + 5\, dt + \int_0^{e^x} 3t^2 + 5\, dt =$$

$$\left(\int_0^{\cos(x)} 3t^2 + 5\, dt\right)' = \left(3\left(\cos(x)\right)^2 + 5\right)\left(\cos(x)\right)' =$$

Regla de la Cadena

$$= \left(3\cos^2(x) + 5\right)\left(-\operatorname{sen}(x)\right) =$$

$$= -\left(3\cos^2(x) + 5\right)\operatorname{sen}(x)$$

$$\left(\int_0^{e^x} 3t^2 + 5\, dt\right)' = \left(3\left(e^x\right)^2 + 5\right) e^x$$

Regla Cadena

Entonces $F'(x) = \left(3(e^x)^2+5\right)e^x - \left[-\left(3\cos^2(x)+5\right)\operatorname{sen}(x)\right] =$

$$= \left(3(e^x)^2+5\right)e^x + \left(3\cos^2(x)+5\right)\operatorname{sen}(x)$$

Fórmula de integración por Partes

Sean $f, g : I \longrightarrow \mathbb{R}$ funciones derivables tales que f' y g' son continuas

Entonces

$$\int f'.g(x)\,dx = f.g - \int f.g'(x)\,dx$$

Se lée: una primitiva de $f'.g$

Se lée: una primitiva de $f.g'$

Ejemplo:

$f(x) = e^x \Rightarrow f'(x) = e^x$

$g(x) = x \Rightarrow g'(x) = 1$

$$\int e^x . x\,dx = e^x . x - \int e^x . 1\,dx = e^x . x - e^x =$$

$$\int e^x \cdot x\, dx \quad \ldots$$

f' g $\qquad$ f g $\qquad$ f g'

$$= \boxed{e^{x}(x-1)}$$

$$\int \operatorname{sen}(x)\cdot x\, dx \underset{\text{F. Integración por Partes}}{=} -\cos(x)\cdot x - \int (-\cos(x))\cdot 1\, dx \overset{*}{=}$$

$\operatorname{sen}(x)\cdot x = f'\cdot g$ donde $f' = \operatorname{sen}(x) \Longrightarrow f = -\cos(x)$

$g = x \Longrightarrow g' = 1$

$$\overset{*}{=} -\cos(x)\cdot x + \int \cos(x)\, dx = -\cos(x)x + \operatorname{sen}(x)$$

¿Porqué es cierta la fórmula de integración por partes?

Para ver que $\displaystyle\int f'g(x)\,dx = fg - \underbrace{\int fg'(x)\,dx}_{\varphi}$

Tenemos que ver que si

$\displaystyle\varphi = \int fg'(x)\,dx$ entonces

$$(f\cdot g - \varphi)' \overset{?}{=} f'g$$

Regla de Leibnitz

$$(fg-\varphi)' = (f\cdot g)' - (\varphi)' = f'\cdot g + f\cdot g' - \varphi' =$$

$$= f'g + fg' - fg' = f'g$$

$$\int \underbrace{\operatorname{sen}(x)}_{f'} \cdot \underbrace{\cos(x)}_{g}\, dx = \underbrace{(-\cos(x))}_{f}\underbrace{(\cos(x))}_{g} - \int \underbrace{(-\cos(x))}_{f}\underbrace{(-\operatorname{sen}(x))}_{g'}\, dx =$$

$f' = \operatorname{sen}(x) \Rightarrow f = -\cos(x)$

$g = \cos(x) \Rightarrow g' = -\operatorname{sen}(x)$

$$= -\cos^2(x) - \int \operatorname{sen}(x)\cos(x)\, dx$$

Entonces

$$2\cdot \int \operatorname{sen}(x)\cdot\cos(x)\, dx = -\cos^2(x)$$

$$\Rightarrow \int \operatorname{sen}(x)\cos(x)\, dx = \frac{-\cos^2(x)}{2}$$

Verifiquemos:

$\left(\quad\right)'$?

$$\left(\frac{-\cos^2(x)}{2}\right)' = \operatorname{sen}(x)\cos(x)$$

$$\left(\frac{-\cos^2(x)}{2}\right)' = \frac{-1}{2}\cdot 2\cdot\cos(x)(-\operatorname{sen}(x)) =$$

$$= -\cos(x)(-\operatorname{sen}(x)) = \cos(x)\operatorname{sen}(x)$$ ✓

InTenTo 1

$$\int \underbrace{x^2}_{f'}\cdot\underbrace{\cos(x)}_{g}\,dx = \underbrace{\frac{x^3}{3}}_{f}\cdot\underbrace{\cos(x)}_{g} - \int \underbrace{\frac{x^3}{3}}_{f}\cdot\underbrace{(-\operatorname{sen}(x))}_{g'}\,dx$$

$f' = x^2 \Rightarrow f = \frac{x^3}{3}$

$g = \cos(x) \Rightarrow g' = -\operatorname{sen}(x)$

InTenTo 2

$$\int \underbrace{x^2}_{g}\,\underbrace{\cos(x)}_{f'}\,dx = \operatorname{sen}(x)x^2 - 2\boxed{\int \operatorname{sen}(x)\cdot x\,dx}$$

$f' = \cos(x) \Rightarrow f = \operatorname{sen}(x)$

$g = x^2 \Rightarrow g' = 2x$

$$\int \underbrace{\operatorname{sen}(x)}_{f'}\,\underbrace{x}_{g}\,dx = -\cos(x).x - \int -\cos(x) =$$ ⊛

$g = x \Rightarrow g' = 1$

$$f' = \operatorname{sen}(x) \Rightarrow f = -\cos(x)$$

$$\overset{*}{=} -\cos(x)\cdot x + \int \cos(x)\,dx = \boxed{-\cos(x)x + \operatorname{sen}(x)}$$

Volvemos al cálculo original:

$$\int x^2 \cos(x) = \operatorname{sen}(x)x^2 - 2\left(-\cos(x)x + \operatorname{sen}(x)\right) =$$

$$= \boxed{\operatorname{sen}(x)x^2 + 2\cos(x)x - 2\operatorname{sen}(x)}$$

Verifiquemos:

$$\left(\operatorname{sen}(x)x^2 + 2\cos(x)x - 2\operatorname{sen}(x)\right)' =$$

$$= \cos(x)x^2 + \cancel{\operatorname{sen}(x)(2x)} + \cancel{2(-\operatorname{sen}(x))x} + \cancel{2\cos(x)}) - \cancel{2\cos(x)} =$$

$$= \boxed{\cos(x)\cdot x^2}$$