Theorem. *Let γ be a piecewise smooth curve in the complex plane. If $\{f_j\}$ is a sequence of continuous complex-valued functions on γ, and if $\{f_j\}$ converges uniformly to f on γ, then $\int_\gamma f_j(z)dz$ converges to $\int_\gamma f(z)dz$.*

The first theorem above on the continuity of a uniform limit has a standard formal proof, which we omit. The second theorem above on the limit of the integrals is an easy consequence of the ML-estimate. Indeed, suppose $\{f_j\}$ converges uniformly to f on γ. Let ε_j be the worst-case estimator for $f_j - f$ on γ, so that $|f_j - f| \le \varepsilon_j$ on γ, and let L be the length of γ. Then the ML-estimate gives

$$\left| \int_\gamma f_j(z)dz - \int_\gamma f(z)dz \right| \;\le\; \varepsilon_j L,$$

and this tends to 0, since the f_j's converge uniformly to f. Hence $\int f_j dz$ tends to $\int f\,dz$.

Now we turn to series of functions. Let $\sum g_j(x)$ be a series of complex-valued functions defined on a set E. The partial sums of the series are the functions

$$S_n(x) \;=\; \sum_{k=0}^{n} g_j(x) \;=\; g_0(x) + g_1(x) + \cdots + g_n(x).$$

We say that the series **converges pointwise** on E if the sequence of partial sums converges pointwise on E, and the series **converges uniformly** on E if the sequence of partial sums converges uniformly on E. The following criterion for uniform convergence of a series of functions is extremely useful. In fact, it is the only test for uniform convergence of series that we will ever need.

Theorem (Weierstrass M-Test). *Suppose $M_k \ge 0$ and $\sum M_k$ converges. If $g_k(x)$ are complex-valued functions on a set E such that $|g_k(x)| \le M_k$ for all $x \in E$, then $\sum g_k(x)$ converges uniformly on E.*

The proof is straightforward. For each fixed x, the estimate for $g_k(x)$ shows that the series $\sum g_k(x)$ is absolutely convergent, and $\sum |g_k(x)| \le \sum M_k$. By the theorem in Section 1, the series $\sum g_k(x)$ converges to some complex number $g(x)$, and by (1.1), $|g(x)| \le \sum |g_k(x)| \le \sum M_k$. The same estimate, applied to the tail of the series, shows that

$$\left| g(x) - S_n(x) \right| \;=\; \left| \sum_{k=n+1}^{\infty} g_k(x) \right| \;\le\; \sum_{k=n+1}^{\infty} M_k\,.$$

If we set $\varepsilon_n = \sum_{k=n+1}^{\infty} M_k$, then $\varepsilon_n \to 0$ as $n \to \infty$, and the estimate shows that the partial sums $S_n(x)$ converge uniformly on E to $g(x)$.