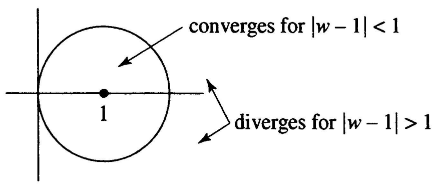
Example. If we integrate the geometric series term by term, we obtain

$$\begin{aligned} -\operatorname{Log}(1-z) &= \int_0^z \frac{d\zeta}{1-\zeta} = \int_0^z \sum_{k=0}^{\infty} \zeta^k \, d\zeta \\ &= \sum_{k=0}^{\infty} \frac{z^{k+1}}{k+1} = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots, \qquad |z| < 1. \end{aligned}$$

Making the substitution $w = 1 - z$, we obtain a series expansion for $\operatorname{Log} w$ centered at $w = 1$,

$$\begin{aligned} \operatorname{Log} w &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (w-1)^k \\ &= (w-1) - \frac{(w-1)^2}{2} + \frac{(w-1)^3}{3} - \frac{(w-1)^4}{4} + \cdots, \qquad |w-1| < 1. \end{aligned}$$



Now we turn to two formulae for determining the radius of convergence of a power series from its coefficients. The first of these is based on the **ratio test**. It is especially convenient for determining the radius of convergence of many series that arise as solutions of linear differential equations.

Theorem. *If $|a_k/a_{k+1}|$ has a limit as $k \to \infty$, either finite or $+\infty$, then the limit is the radius of convergence R of $\sum a_k z^k$,*

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|.$$

To see this, let $L = \lim |a_k/a_{k+1}|$. If $r < L$, then $|a_k/a_{k+1}| > r$ eventually, say for $k \geq N$. Then $|a_k| > r|a_{k+1}|$ for $k \geq N$, and

$$|a_N| r^N \geq |a_{N+1}| r^{N+1} \geq |a_{N+2}| r^{N+2} \geq \cdots.$$

Hence the sequence $|a_k| r^k$ is bounded. From the definition of R we have $r \leq R$, and since $r < L$ is arbitrary, we also have $L \leq R$.

Suppose next that $s > L$. Then $|a_k/a_{k+1}| < s$ eventually, say for $k \geq N$. Then $|a_k| < s|a_{k+1}|$ for $k \geq N$, and

$$|a_N| s^N \leq |a_{N+1}| s^{N+1} \leq |a_{N+2}| s^{N+2} \leq \cdots.$$

Hence the terms $a_k z^k$ do not converge to 0 for $|z| \geq s$, so that the series does not converge, and $s \geq R$. Since $s > L$ is arbitrary, we also have $L \geq R$. We conclude that $L = R$.

Example. For the series $\sum kz^k$, the ratio test gives

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{k}{k+1} \to 1.$$

Hence the radius of convergence is $R = 1$.

Example. For the series $\sum \frac{z^k}{k!}$, the ratio test gives

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{(k+1)!}{k!} = k+1 \to +\infty.$$

Hence the radius of convergence is $R = +\infty$.

The second formula is based on the **root test**.

Theorem. *If $\sqrt[k]{|a_k|}$ has a limit as $k \to \infty$, either finite or $+\infty$, then the radius of convergence of $\sum a_k z^k$ is given by*

$$R = \frac{1}{\lim \sqrt[k]{|a_k|}}. \tag{3.3}$$

If $r > 1/\lim \sqrt[k]{|a_k|}$, then $\sqrt[k]{|a_k|}\, r > 1$ eventually, so that $|a_k|r^k > 1$ eventually, the terms of the series $\sum a_k z^k$ do not converge to 0 for $|z| = r$, and $r \geq R$. On the other hand, if $r < 1/\lim \sqrt[k]{|a_k|}$, then $\sqrt[k]{|a_k|}\, r < 1$ eventually, so that $|a_k|r^k < 1$ eventually, the sequence $|a_k|r^k < 1$ is bounded, and from the definition of R we have $r \leq R$. It follows that (3.3) holds.

Example. For the series $\sum kz^k$, the root test gives

$$R = 1/\lim \sqrt[k]{k} = 1.$$

There is a more general form of the formula (3.3), called the **Cauchy-Hadamard formula**, that gives the radius of convergence for *any* power series in terms of a lim sup. Recall (Section II.1) that the lim sup of a sequence $\{s_n\}$ is characterized as the number S, $-\infty \leq S \leq +\infty$, with the property that if $t > S$, then only finitely many terms of the sequence satisfy $s_n > t$, while if $t < S$, then infinitely many terms of the sequence satisfy $s_n > t$. If the sequence s_n has a limit, then the lim sup of the sequence coincides with the limit. However, every sequence has a lim sup. The Cauchy-Hadamard formula is obtained simply by replacing the limit in (3.3) by a lim sup,

$$R = \frac{1}{\limsup \sqrt[k]{|a_k|}}. \tag{3.4}$$