8. Show that $\sum \frac{z^k}{k^2}$ converges uniformly for $|z| < 1$.

9. Show that $\sum \frac{z^k}{k}$ does not converge uniformly for $|z| < 1$.

10. Show that if a sequence of functions $\{f_k(x)\}$ converges uniformly on E_j for $1 \leq j \leq n$, then the sequence converges uniformly on the union $E = E_1 \cup E_2 \cup \cdots \cup E_n$.

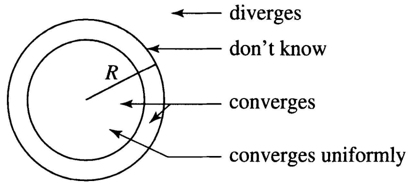
11. Suppose that E is a bounded subset of a domain $D \subset \mathbb{C}$ at a positive distance from the boundary of D, that is, there is $\delta > 0$ such that $|z - w| \geq \delta$ for all $z \in E$ and $w \in \mathbb{C} \backslash D$. Show that E can be covered by a finite number of closed disks contained in D. *Hint.* Consider all closed disks with centers at points $(m + ni)\delta/10$ and radius $\delta/10$ that meet E.

12. Let $f(z)$ be analytic on a domain D, and suppose $|f(z)| \leq M$ for all $z \in D$. Show that for each $\delta > 0$ and $m \geq 1$, $|f^{(m)}(z)| \leq m!M/\delta^m$ for all $z \in D$ whose distance from ∂D is at least δ. Use this to show that if $\{f_k(z)\}$ is a sequence of analytic functions on D that converges uniformly to $f(z)$ on D, then for each m the derivatives $f_k^{(m)}(z)$ converge uniformly to $f^{(m)}(z)$ on each subset of D at a positive distance from ∂D.

3. Power Series

A **power series** (centered at z_0) is a series of the form $\sum_{k=0}^{\infty} a_k(z - z_0)^k$. By making a change of variable $w = z - z_0$, we can always reduce to the case of power series centered at $z = 0$. The main result on convergence of power series is the following.

Theorem. *Let $\sum a_k z^k$ be a power series. Then there is R, $0 \leq R \leq +\infty$, such that $\sum a_k z^k$ converges absolutely if $|z| < R$, and $\sum a_k z^k$ does not converge if $|z| > R$. For each fixed r satisfying $r < R$, the series $\sum a_k z^k$ converges uniformly for $|z| \leq r$.*



We call R the **radius of convergence** of the series $\sum a_k z^k$. The radius of convergence depends only on the tail of the series. If we alter a finite number of coefficients of the series, the radius of convergence remains the same.

For the general case of a power series $\sum a_k(z - z_0)^k$, the domain of convergence is a disk $|z - z_0| < R$. The series diverges if $|z - z_0| > R$, and anything can happen when $|z - z_0| = R$.

For the proof of the theorem, note first that if the sequence $|a_k|r^k$ is bounded for some value $r = r_0$, then it is bounded for all r satisfying $0 \le r < r_0$. We define R, $0 \le R \le +\infty$, to be the supremum of the r's such that $|a_k|r^k$ is bounded. Thus $|a_k|r^k$ is bounded if $r < R$, while if $r > R$, then there is a sequence of terms with $|a_{k_j}|r^{k_j} \to +\infty$. In the borderline case $r = R$, anything can happen. The sequence $|a_k|R^k$ might be bounded and it might not.

If $|z| > R$, then the terms $a_k z^k$ do not tend to 0, so that the series does not converge. On the other hand, suppose $r < R$. Choose s such that $r < s < R$. Then the sequence $|a_k|s^k$ is bounded, say $|a_k|s^k \le C$ for $k \ge 0$. If $|z| \le r$, then

$$|a_k z^k| \le |a_k|r^k = |a_k|s^k\left(\frac{r}{s}\right)^k \le C\left(\frac{r}{s}\right)^k .$$

Set $M_k = C(r/s)^k$. Since $\sum M_k$ converges, the Weierstrass M-test applies, and the series $\sum a_k z^k$ converges uniformly for $|z| \le r$, and also absolutely for each z. This proves the theorem.

Example. The geometric series $\sum z^k$ has radius of convergence $R = 1$. The series does not converge on the boundary circle $|z| = 1$, since the terms do not tend to 0.

Example. The power series $\sum z^k/k^2$ converges uniformly for $|z| \le 1$. This follows from the Weierstrass M-test, with majorants $M_k = 1/k^2$. On the other hand, if $r > 1$, then $r^k/k^2 \to \infty$ as $k \to \infty$. Thus the series does not converge for $|z| > 1$, and the radius of convergence of the series is $R = 1$.

Example. The series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k} = 1 - \frac{z^2}{2} + \frac{z^4}{2^2} - \frac{z^6}{2^3} + \cdots \tag{3.1}$$

becomes a geometric series if we set $w = -z^2/2$,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} z^{2k} = \sum_{k=0}^{\infty} w^k .$$

The series converges precisely when $|w| < 1$, that is, when $|z^2| < 2$. The radius of convergence is thus $R = \sqrt{2}$. The series converges to $1/(1 - w) = 2/(2 - z^2)$.