Def:

$\operatorname{ord}(f) > \operatorname{ord}(g)$ en $\boxed{+\infty}$ si

$$\lim_{x\to \boxed{+\infty}} \frac{f(x)}{g(x)} = +\infty \iff \lim_{x\to+\infty} \frac{g(x)}{f(x)} = 0^+$$

$$\boxed{\operatorname{ord}(\log(x)) < \operatorname{ord}(x^\alpha) < \operatorname{ord}(e^x) \quad \text{en } \boxed{+\infty}}$$
$$\forall \alpha > 0$$

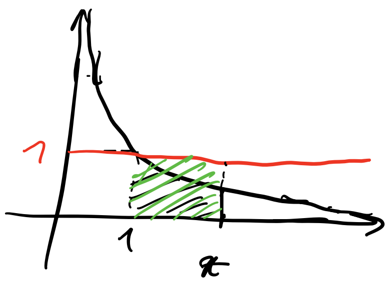
Primero vamos a ver que

(*) $\operatorname{ord}(\log(x)) < \operatorname{ord}(x^\alpha)$ en $\boxed{+\infty}$ $\boxed{\forall \alpha > 1}$

Antes de probar esto, vamos a probar que

$$0 < \boxed{\frac{\log(x)}{x} < 1} \quad \forall x > 1$$

$$\log(x) = \int_1^x \frac{1}{t}\,dt$$



$$\frac{1}{t} \le 1 \quad \forall t \geq 1 \underset{\text{Prop. Monotonía}}{\Longrightarrow} \underbrace{\int_1^x \frac{1}{t}\,dt}_{\log(x)} \le \int_1^x 1\,dt = x-1 \quad \forall x>1$$

$$\Rightarrow \log(x) \le x-1 < x \quad \forall x>1$$

$$\Rightarrow \boxed{\frac{\log(x)}{x} < 1} \quad \forall x > 1$$

$\boxed{\alpha > 1}$

$$\lim_{x\to+\infty} \frac{\log(x)}{x^{\alpha}} = \lim_{x\to+\infty} \overbrace{\frac{\log(x)}{x}}^{\text{ACOTADO POR } (*)} \cdot \overbrace{\frac{1}{\underbrace{x^{\overbrace{\alpha-1}^{>0}}}_{\to+\infty}}}^{\to 0}$$

$$\underset{\text{ACOTADO} \times 0}{\Longrightarrow} \lim_{x\to+\infty} \frac{\log(x)}{x^{\alpha}} = 0 \quad \checkmark$$

$$\Longrightarrow \text{ord}(\log(x)) < \text{ord}(x^{\alpha}) \qquad (\alpha > 1)$$

CAMBIO DE VARIABLE PARA LÍMITES

$\lim_{u\to+\infty} h(u) = +\infty$, h continua

$$\Longrightarrow \boxed{\lim_{x\to+\infty} f(x) = \lim_{u\to+\infty} f(h(u))} \qquad \text{I}$$

$$\lim_{x\to 0^+} f(x) = \lim_{u\to+\infty} f\left(\frac{1}{u}\right) \qquad \text{II}$$

Veamos ahora que $\text{ord}(\log(x)) < \text{ord}(x^{\alpha})$ $\alpha > 0$

en $\boxed{+\infty}$

$$\lim_{x\to+\infty} \frac{\log(x)}{x^{\alpha}} \underset{\text{C.V I}}{=} \lim_{u\to+\infty} \frac{\log\left(u^{\frac{2}{\alpha}}\right)}{\left(u^{\frac{2}{\alpha}}\right)^{\alpha}} =$$

$h(u) = u^{\frac{2}{\alpha}}$

Como $\frac{2}{\alpha} > 0$; $\lim_{u \to +\infty} u^{\frac{2}{\alpha}} = +\infty$

$$= \lim_{u \to +\infty} \frac{\frac{2}{\alpha} \log(u)}{u^2} = 0$$

$\to 0$

$$\boxed{\begin{array}{l} \log(a^b) = b \cdot \log(a) \\ (a^b)^c = a^{(b \cdot c)} \end{array}}$$

$$\Longrightarrow \text{ord}(\log(x)) < \text{ord}(x^{\alpha}) \quad \forall \alpha > 0 \quad \text{en } \boxed{+\infty}$$

Veamos que $\text{ord}(x^{\alpha}) < \text{ord}(e^x)$ en $\boxed{+\infty}$

$\boxed{\alpha > 0}$

$\boxed{\alpha > 0}$

$$\lim_{x \to +\infty} \frac{x^{\alpha}}{e^x} \underset{\substack{\text{CV I} \\ x = \log(u)}}{=} \lim_{u \to +\infty} \frac{(\log(u))^{\alpha}}{e^{\log(u)}} =$$

$$= \lim_{u \to +\infty} \frac{\log(u)^{\alpha}}{u} = \lim_{u \to +\infty} \left(\frac{\log(u)}{u^{\frac{1}{\alpha}}} \right)^{\alpha} = 0$$

$\to 0$

$$\frac{\log(u)^{\alpha}}{u} = \left(\frac{\log(u)^{\alpha}}{u} \right)^{\frac{1}{\alpha} \cdot \alpha} = \left(\left(\frac{\log(u)^{\alpha}}{u} \right)^{\frac{1}{\alpha}} \right)^{\alpha} =$$

$\stackrel{!}{=}$

$$= \left(\frac{\left(\log(u)^{\alpha}\right)^{\frac{1}{\alpha}}}{u^{\frac{1}{\alpha}}}\right)^{\alpha} = \left(\frac{\log(u)}{u^{\frac{1}{\alpha}}}\right)^{\alpha}$$

RESOLUVAMOS OTRA INDETERMINACIÓN

$$\lim_{x\to 0^+} \underbrace{x^{\alpha}}_{\to 0}\ \underbrace{\log(x)}_{\to -\infty} \quad ; \quad \alpha > 0$$

INDETERMINACIÓN $0.(-\infty)$

$$\lim_{x\to 0^+} x^{\alpha}\log(x) \underset{x=\frac{1}{u}}{=} \lim_{u\to+\infty} \left(\frac{1}{u}\right)^{\alpha}\log(u^{-1}) =$$

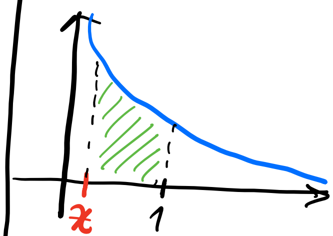
$$= \lim_{u\to+\infty} \frac{1}{u^{\alpha}}(-1).\log(u) =$$

$$= -\lim_{u\to+\infty} \frac{\log(u)}{u^{\alpha}} = 0 .$$

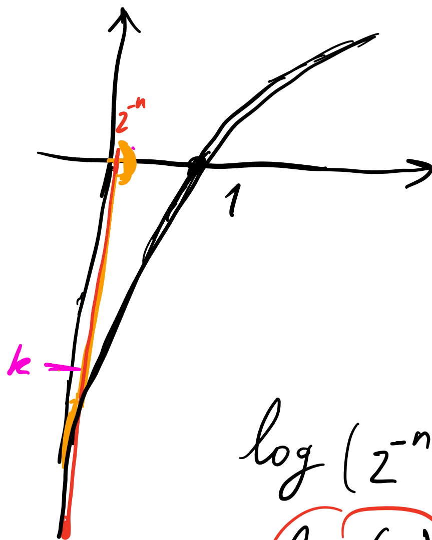
$$\Rightarrow \lim_{x\to 0^+} x^{\alpha}.\log(x) = 0 \quad \forall \alpha > 0$$

Paréntesis:

¿Porqué $\log(x) \xrightarrow[x\to 0^+]{} -\infty$?



$$\log(x) = \int_1^x \frac{1}{t}\,dt = -\left(\int_x^1 \frac{1}{t}\,dt\right)$$



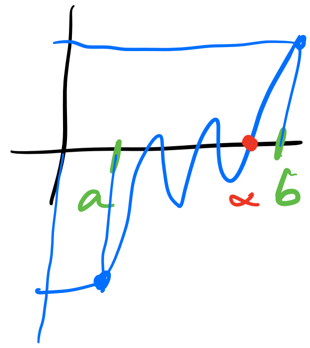
$$\log(2^{-n}) = \underbrace{-n.\underbrace{\log(2)}_{>0}}$$

TEOREMA DE BOLZANO

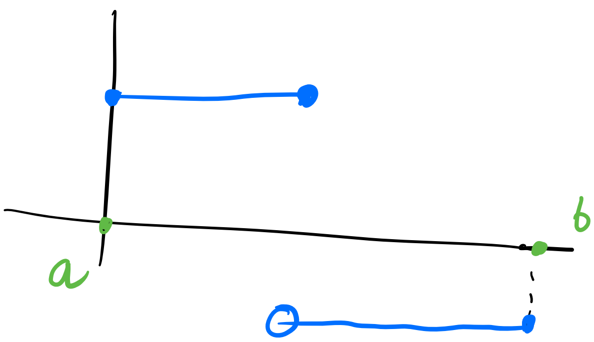
Sea $f:[a,b]\longrightarrow\mathbb{R}$ $\boxed{\text{continua}}$

supongamos que $f(a)\cdot f(b)<0$

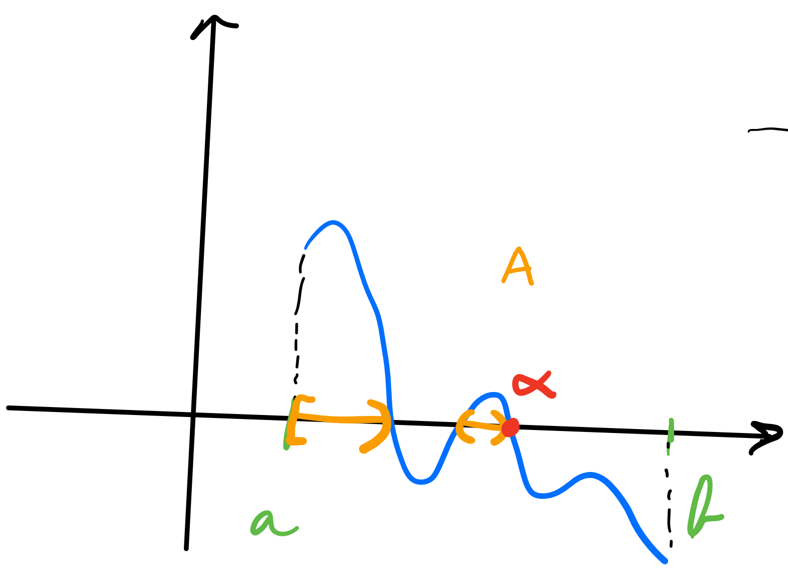
Entonces, $\exists\ \alpha\in(a,b)\ /\ f(\alpha)=0$

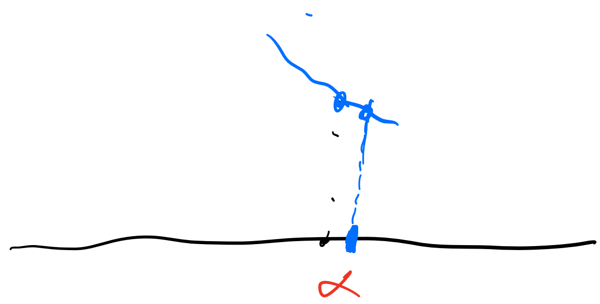


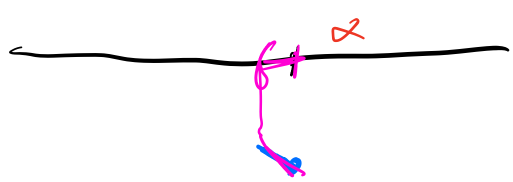
La hipótesis f continua es necesaria para que se cumpla la tesis.



Idea de la prueba:







$A = \{x \in [a,b] : f(x) > 0\}$

$\alpha := \sup(A)$

Vamos a probar $f(\alpha) = 0$

¿Cómo? por absurdo.

suponiendo que no, es decir $f(\alpha) \neq 0$

Tenemos dos casos: $f(\alpha) > 0 \rightsquigarrow$ Absurdo

$f(\alpha) < 0 \rightsquigarrow$ Absurdo