Linealidad de la integral

Si $f, g : [a,b] \longrightarrow \mathbb{R}$ son integrables.
Entonces $f+g$ es integrable

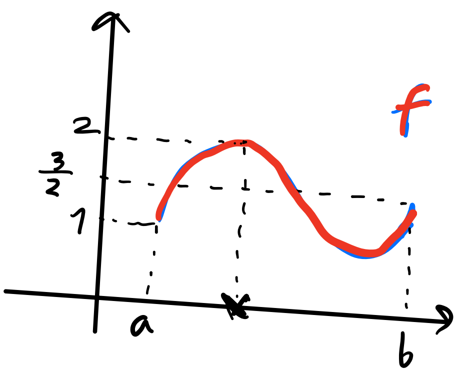
$$\boxed{\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx}$$

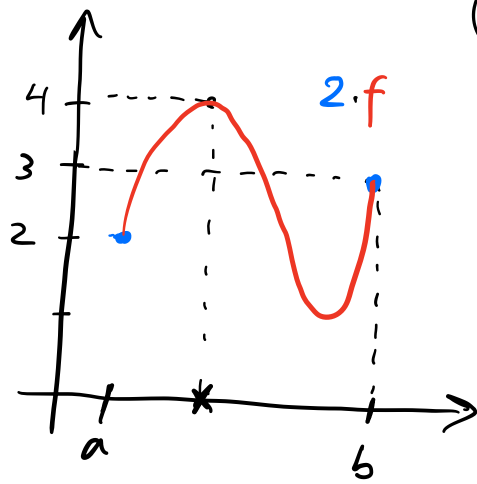
Además si $\alpha \in \mathbb{R}$, entonces αf es integrable y

$$\boxed{\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx}$$

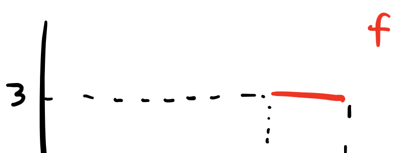
¿Porqué vale la linealidad?

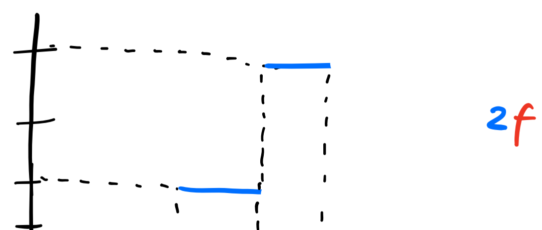
$(2f)(a) = 2 \cdot f(a)$





Ej sencillo:

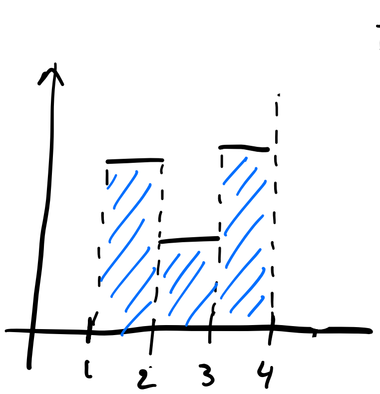


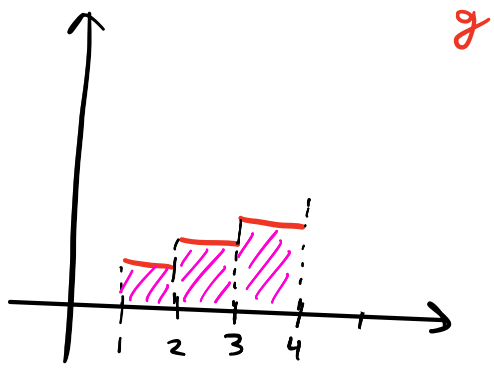


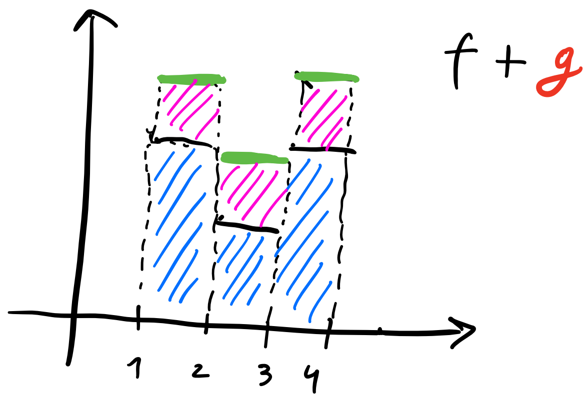
$$\int_1^4 f = 1 + 2 + 3 = 6$$

$$\int_1^4 2f = 2 + 4 + 6 = 2\int f$$

<u>Para la suma</u>, un ejemplo sencillo

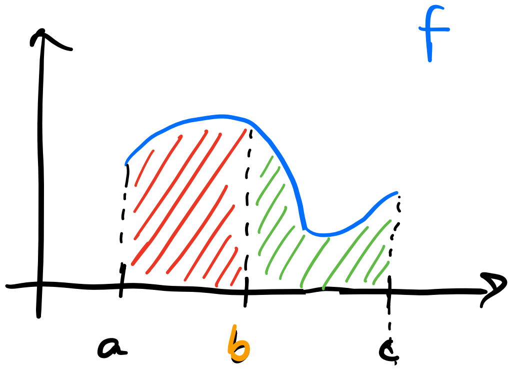






$$\int_1^4 f(x)\,dx + \int_1^4 g(x)\,dx = \int_1^4 (f+g)(x)\,dx$$

<u>Linealidad respecto del intervalo</u>



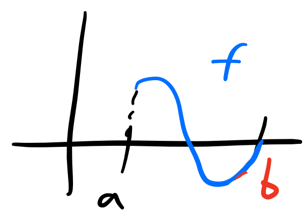
Sean $a < b < c$ números reales y $f : [a,c] \longrightarrow \mathbb{R}$ integrable, Entonces $f_{|[a,b]}$ y $f_{|[b,c]}$ son integrables y se tiene

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$$

Extensión de la definición de integral

Si $a < b$; definimos

$$\boxed{\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx}$$



$$\int_b^a f$$

Ej:

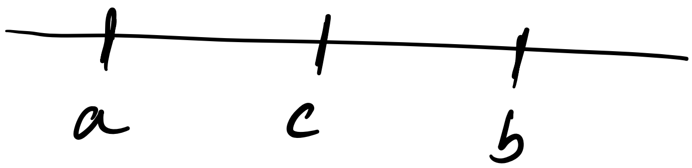
Si $\int_1^4 f(x)\,dx = 6 \Rightarrow \int_4^1 f(x)\,dx = -6$

$$\int_a^a f(x)\,dx = 0$$

Propiedad: si $a, b, c \in \mathbb{R}$ (no necesariamente en orden creciente)

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

Deduzcamos la propiedad de la linealidad respecto del intervalo, en un caso.



hay que ver que

$$\int_a^c f(x)\,dx = \left(\int_a^b f(x)\,dx\right) + \left(\int_b^c f(x)\,dx\right)$$

Por linealidad respecto del intervalo; como $a<c<b$;

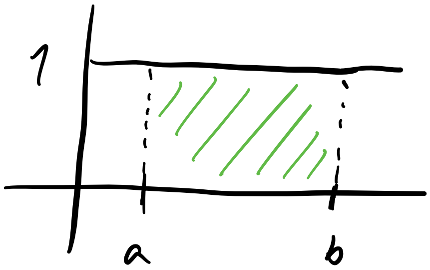
$$\left(\int_a^b f(x)\,dx\right) = \left(\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right)$$

Por otro lado, $\left(\int_b^c f(x)\,dx\right) = \left(-\int_c^b f(x)\,dx\right)$

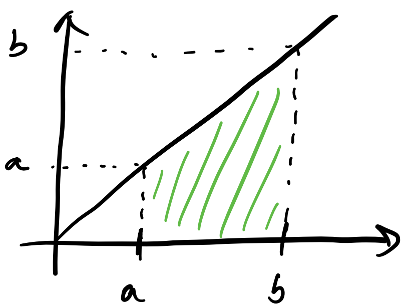
Entonces

$$\left(\int_a^b f\right) + \int_b^c f = \left(\int_a^c f + \cancel{\int_c^b f}\right) + \left(-\cancel{\int_c^b f}\right) = \int_a^c f$$

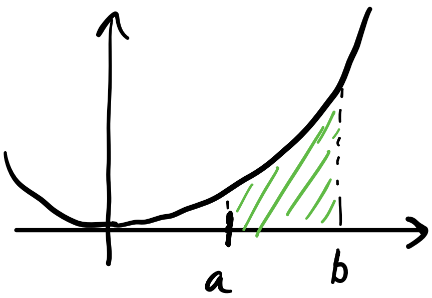
Spoiler:



$$\int_a^b 1\,dx = b-a$$



$$\int_a^b x\,dx = \frac{b^2-a^2}{2}$$



$$\int_a^b x^2\,dx = \frac{b^3-a^3}{3}$$

$$\int_a^b x^n = \frac{b^{n+1}-a^{n+1}}{n+1}$$

después del parcial

$$\int_1^3 3x^2+5x-7\,dx = \int_1^3 3x^2\,dx + \int_1^3 5x\,dx + \int_1^3 (-7)\,dx =$$

linealidad de la integral

$$= 3\int_1^3 x^2\,dx + 5\int_1^3 x\,dx + (-7)\int_1^3 1\,dx =$$

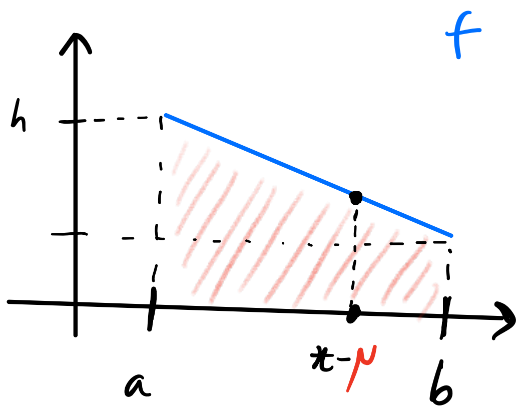
linealidad de la integral

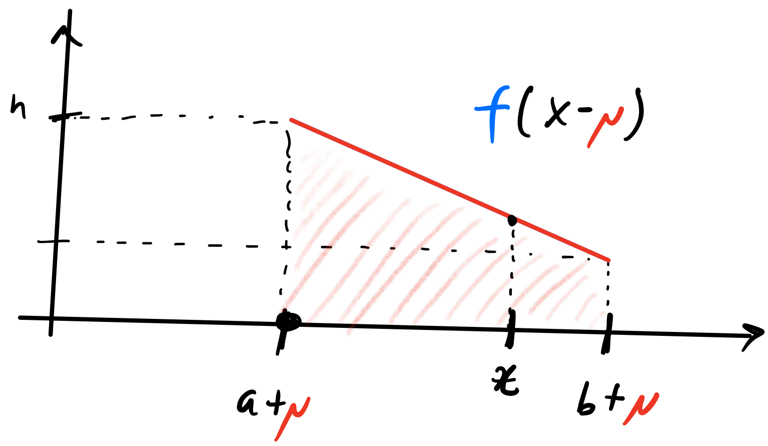
$$= 3\left(\frac{3^3-1^3}{3}\right)+5\left(\frac{3^2-1^2}{2}\right)+(-7)(3-1)=\boxed{3\frac{26}{3}+5\frac{8}{2}-14}$$

sustituimos

spoiler

Fórmulas de cambio de variable lineal



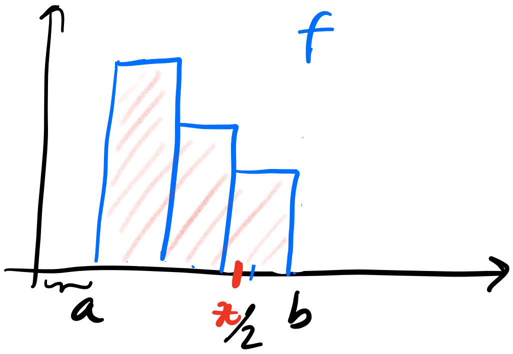


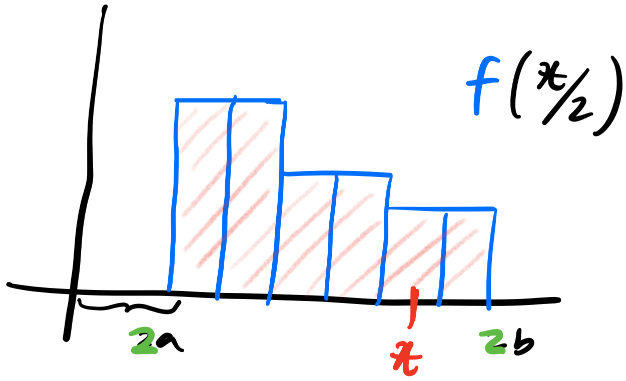
$$\boxed{\int_a^b f(x)\,dx = \int_{a+\mu}^{b+\mu} f(x-\mu)\,dx}$$

Ej: $$\int_3^5 (x-7)^2\,dx = \int_{3-7}^{5-7} x^2\,dx = \int_{-4}^{-2} x^2\,dx =$$

$$= \frac{(-2)^3-(-4)^3}{3}$$

Cambio de variable por una dilatación

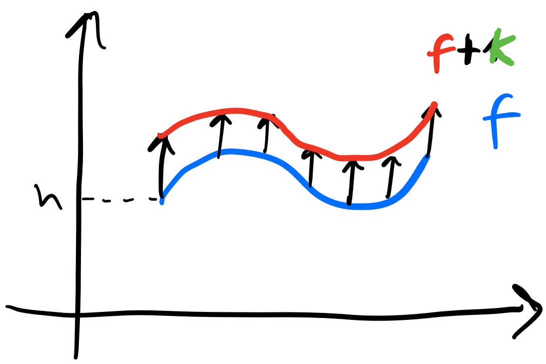




$$\boxed{\lambda \int_a^b f(x)\,dx = \int_{\lambda a}^{\lambda b} f\left(x/\lambda\right) dx}$$

$$\boxed{\lambda \int_a^b f(x)\,dx = \int_{\lambda a+\mu}^{\lambda b+\mu} f\left(\frac{x-\mu}{\lambda}\right) dx}$$

¿Qué pasa si trasladamos el gráfico en la dirección del eje y?



$$\int_a^b f+k\,dx = \int_a^b f\,dx + \int_a^b k =$$

$$= \int_a^b f(x)\,dx + k(b-a)$$