Regla del Jacobiano y cambio de variables

① $Y = g(X)$ con g monótona y con derivada no nula

Entonces $\displaystyle f_Y(y) = f_X(x) \frac{1}{|g'(x)|} \Big|_{x = g^{-1}(y)}$.

dem: $F_Y(y) = P(Y \le y)$

$= P(g(X) \le y)$

supongo $g \nearrow$

$= P(X \le g^{-1}(y))$

$= F_X(g^{-1}(y))$

si $g \searrow$

$P(X \ge g^{-1}(y))$

$= 1 - P(X \le g^{-1}(y))$

Derivo: $\displaystyle \frac{-f_X(g^{-1}(y))}{g'(g^{-1}(y))}$

Derivo $f_Y(y) = F'_X(g^{-1}(y)) \, (g^{-1})'(y)$

$\displaystyle = f_X(g^{-1}(y)) \frac{1}{g'(g^{-1}(y))}$

Ejemplo: $Y = aX + b$. con $a > 0$.

$\displaystyle \Rightarrow f_Y(y) = f_X\left(\frac{y-b}{a}\right) \frac{1}{a}$

② $f_Y(y) = f_X(g^{-1}(y)) \, |J_{g^{-1}}(y)|$ si $Y = g(X)$ con g invertible.

$P(Y \in B) = \int_B \boxed{f_Y(y)} = P(g(X) \in B)$

$= P(X \in g^{-1}(B)) = \int_{g^{-1}(B)} f_X(x) \, dx$.

$x = g^{-1}(y)$

$= \int_B \boxed{f_X(g^{-1}(y)) \, |J_{g^{-1}}(y)|} \, dy$

Ejemplo: A invertible e $Y = AX$ $\quad f_Y(y) = f_X(A^{-1}y) \, |\det A^{-1}|$

Si g no es monótona:

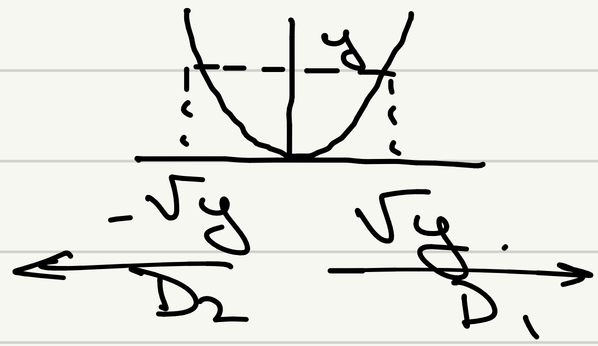
$$f_Y(y) = \sum_{i : y \in g_i(D_i)} f_X\left(g_i^{-1}(y)\right) \left| \frac{d g_i^{-1}(y)}{dy} \right|$$

donde $\bigcup_i D_i = D$ $\quad g_i = g|_{D_i}$ monótona, diferenciable con derivada no nula

1) Sea $X \sim N(0,1)$ y considero $Y = X^2$

$g(x) = x^2 \qquad g_1^{-1}(y) = \sqrt{y}$

$\qquad\qquad\qquad g_2^{-1}(y) = -\sqrt{y}$



si $y > 0$

$$f_Y(y) = f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \left| \frac{-1}{2\sqrt{y}} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{y})^2} \frac{1}{2\sqrt{y}} + \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-\sqrt{y})^2}$$

$$= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} & , y \geq 0 \\ 0 & , y < 0 \end{cases}$$

$\Rightarrow \boxed{Y \sim \chi^2(1)}$

2) Sea $X \sim N(0,1)$ e $Y = 2X + 3$.

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-3}{2}\right)^2}$$

$\Rightarrow \boxed{Y \sim N(3, 2^2)}$

3) $X \sim U[-1,1]$ $\quad Y = e^X$ $\quad f_X(x) = \begin{cases} \frac{1}{2}, & x \in [-1,1] \\ 0, & \text{sino} \end{cases}$

$g^{-1}(y) = \ln(y)$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \begin{cases} \frac{1}{2} \cdot \frac{1}{y} & \text{si } g^{-1}(y) \in [-1,1] \\ 0 & \text{sino} \end{cases}$$

$$= \begin{cases} \frac{1}{2y} & \text{si } y \in [1/e, e] \\ 0 & \text{si no} \end{cases}$$

4) $X \sim U[-1,1]$ $\quad Y = X^2$

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$= \frac{\sqrt{y}+1}{2} - \frac{-\sqrt{y}+1}{2} = \sqrt{y}$$

$$\begin{cases} \sqrt{y}, & 0 < y < 1 \\ 0, & y \leq 0 \\ 1, & y \geq 1. \end{cases}$$