## SECTION IV

NONLINEAR SYSTEMS ANALYSIS

## PHASE-PLANE ANALYSIS

The objective of this chapter is to introduce the student to phase-plane analysis, which is one of the most important techniques for studying the behavior of nonlinear systems. After studying this chapter, the student should be able to:

- Use eigenvalues and eigenvectors of the Jacobian matrix to characterize the phaseplane bchavior
- Predict the phase-plane behavior close to an equilibrium point, based on the linearized model at that equilibrium point
- Predict qualitatively the phase-plane behavior of the nonlinear system, when there are multiple equilibrium points

The major sections of this chapter are:

### 13.1 Background

13.2 Linear System Examples
13.3 Generalization of Phase-Plane Behavior
13.4 Nonlinear Systems

### 13.1 BACKGROUND

Techniques to find the transient (time domain) behavior of linear state-space models were discussed in Chapter 5 . Recall that the response characteristics (relative speed of response) for unforced systems were dependent on the initial conditions. Eigenvalaeleigenvector analysis allowed us to predict the fast and slow (or stable and unstable) initial conditions. If we ploted the transient responses based on a number of initial conditions, there would soon be an overwhelming number of curves on the transient response plots. Another way of obtaining a feel for the effect of initial conditions is to use a phaseplane plot. A phase-plane plot for a wo-state variable system consists of curves of one-state variable versus the state variable $\left(x_{1}(t)\right.$ versus $x_{2}(t)$, where cach curve is based on a different initial condition. A phase-space plot can also be made for three-state variables. where each curve in 3 -space is based on a different initial condition.

Phase-plane analysis is one of the most important techniques for studying the behavior of nonlinear systems, since there is usually no analytical solution for a nonlinear system. It is obviously important to understand phase-plane analysis for linear systems before covering nonlinear systems. Section 13.2 discusses the phase-plane behavior of linear systems and Section 13.3 covers nonlinear systems.

### 13.2 LINEAR SYSTEM EXAMPLES

Nonlinear systems often have multiple steady-state solutions (see Modules 8 and 9 for examples). Phase-plane analysis of nonlinear systems provides an understanding of which steady-state solution that a particular set of intital conditions will converge to. The local behavior (close to one of the steady-state solutions) an be understood from a linear phase-plane analysis of the particular steady-state solution (equilibrium point).

In this section we show the different types of phase-plane behavior that can be exbibited by linear systems. The phase-plane analysis approach will be shown by way of a number of examples.

## EXAMPLE 13.1 A Stable Equilibrium Point (Node Sink)

Consider the system of equations:

$$
\begin{align*}
& \dot{x}_{1}=-x_{1}  \tag{13.1}\\
& \dot{x}_{2}=-4 x_{2} \tag{1.3.2}
\end{align*}
$$

The reader should find that the solution to (13.1) and (13.2) is:

$$
\begin{align*}
& x_{1}(t)=x_{10} e^{\cdot t}  \tag{13.3}\\
& x_{2}(t)=x_{2 o} e^{-4 t} \tag{13.4}
\end{align*}
$$

where $x_{1}$, and $x_{2}$, are the initial conditions for $x_{1}$ and $x_{2}$. We could plot $x_{1}$ and $x_{2}$ as a function of time for a large number of intial conditions (requiring a large number of time domain plots), but the same information is contained on a phase-plane plot as shown in Figure 13.1. Each curve corresponds to a different initial condition. Notice that the solutions converge to (0,0) for all initial conditions. The point $(0,0)$ is a stable equilibrium point for the system of equations (13.1) and (13.2)-the plot shown in Figure 13.1 is often called a stable node.


FIGURE 13.1 Phase-plane plot for Example 13.1. The point $x^{T}=(0,0)$ is a stable node.

## EXAMPLE 13.2 An Unstable Equilibrium Point (Saddle)

Consider the system of equations:

$$
\begin{align*}
& \dot{x}_{1}=-x_{1}  \tag{13.5}\\
& \dot{x}_{2}=4 x_{2} \tag{13.6}
\end{align*}
$$

The student should find that the solution to (13.5) and (13.6) is:

$$
\begin{align*}
& x_{1}(t)=x_{10} e^{-t}  \tag{13.7}\\
& x_{2}(t)=x_{2 o} e^{4 t} \tag{13.8}
\end{align*}
$$

The phase-plane plot is shown in Figure 13.2. If the inital condition for the $x_{2}$ state varable was 0 , then a trajectory that reached the origin could be obtaned. Notice if the initial condition $x_{20}$ is just slightly different than zero, then the soletion will always leave the origin. The origin is an unstable equilibrium point, and the trajectories shown in Figure 13.2 represent a sadde point.

The $x_{1}$ axis represents a stable subspace and the $x_{2}$ axis represents an unstable subspace for this problem. The term saddle can be understood if you view the $x_{1}$ axis as the line (ridge) between the "hom" and rear of a saddle. A ball stanting at the bom could conceptually roll down the saddle and remain exactly on the ridge between the hom and the rear of the sadde. In practice, a smatl perturbation from the ridge would cause the ball to begin rolling to one "stirup" or the other. Similarly, a small perturbation in the initial condition from the $x_{1}$ axis in Example 13.2 would cause the solution to diverge in the unstable direction.


FIGURE 13.2 Phase-plane plot for Example 13.2. The point $x^{T}=(0,0)$ is a saddle point.

Figures 13.1 and 13.2 clearly show the idea of separatices. A separatrix is a line in the phase-plane that is not crossed by any trajectory. In Figure 13.1 the separatices are the coordinate axes. A trajectory that stanted in any quadrant stayed in that quadraut. This is because the eigenvectors are the coordinate axes. Similar behavior is observed in Figus 13.2, except that the $x_{1}$ coordinate axis is unstable.

Solving the equations for Examples 13.1 and 13.2 and the phase-plane trajectories were straight-forward and obvious, because the eigenvectors where simply the coortinate axes. In general, eigenvalue/eigenvector analysis must be used to determine the stable and unstable "subspaces." The eigenvectors are the separatices in the general case.

Example 13.3 shows how eigenvalue/eigenvector analysis is used to find the stable and unstable subspaces, and to define the separatices.

## EXAMPLE 13.3 AnotherSaddle Point Problem

Consider the following system of equations:

$$
\begin{align*}
& \dot{x}_{1}=2 x_{1}+x_{2}  \tag{13.9}\\
& \dot{x}_{2}=2 x_{1}-x_{2} \tag{13.10}
\end{align*}
$$

Using standard state-space notation:

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{A} \mathbf{x} \tag{13.11}
\end{equation*}
$$

The Jacobian matrix is:

$$
A=\left|\begin{array}{cc}
2 & 1 \\
2 & -1
\end{array}\right|
$$

the eigenvalues are:

$$
\lambda_{1}=-1.5616 \quad \lambda_{2}=2.5616
$$

and the eigenvectors are:

$$
\xi_{1}=\left[\begin{array}{c}
0.2703 \\
-0.9628
\end{array}\right] \quad \xi_{2}=\left[\begin{array}{c}
0.8719 \\
0.4896
\end{array}\right]
$$

Since $\lambda_{1}<0, \xi_{1}$ is a stable subspace; also, since $\lambda_{2}>0, \ddot{\xi}_{2}$ is at unstable subspace. A plot of the stable and unstable subspaces is shown in Figure 13.3. These eigenvectors also define the separatices that determine the chatacteristic bebavior of the state trajectories.


FIGURE 13.3 Stable and unstable subspaces for Example 13.3.

The time domain solution to (13.11) is:

$$
\begin{equation*}
\mathbf{x}(t)=e^{i t} \mathbf{x}(0) \tag{13.12}
\end{equation*}
$$

which is often solved as (see Chapter 5):

$$
\begin{equation*}
\mathbf{x}(t)=\mathbf{V} e^{A} \mathbf{V}^{-1} x(0) \tag{13.13}
\end{equation*}
$$

which yields the following solution for this system:

$$
\mathbf{x}(i)=\left[\begin{array}{cc}
0.2703 & 0.8719  \tag{13.14}\\
-0.9628 & 0.4896
\end{array}\right]\left[\begin{array}{cc}
e^{1.5616} & 0 \\
0 & e^{2.5616 t}
\end{array}\right]\left[\begin{array}{cc}
0.5038 & -0.8972 \\
0.9907 & 0.2782
\end{array}\right] \mathbf{x}(0)
$$

Recall that the solution to (13.14), if $x(0)=\xi_{1}$, is $x(I)=\xi_{1} e^{\lambda i t}$, so

$$
\text { if } \mathbf{x}(0)=\left[\left.\begin{array}{c}
0.2703 \\
-0.9628
\end{array} \right\rvert\, \text { then } \quad \mathbf{x}(t)=\left[\begin{array}{c}
0.2703 \\
-0.9628
\end{array}\right] e^{. .5616 t}\right.
$$

The phase-plane plot is shown in Figure 13.4, where the separatrices clearly define the phateplane bebavior.


FIGURE 13.4 Phase-plane plot for Example 13.3.

The previous examples were for systems that exhibited stable node or saddle point behavior. In either case, the eigenvalues and cigenvectors where real. Another type of behavior that can occur is a spiral focus (either stable or unstable), which has complex eigenvalues and eigenvectors. Example 13.4 is an unstable focus.

## EXAMPLE 13.4 Unstable Focus (Spiral Source)

Consider the following system of equations:

$$
\begin{align*}
& \dot{x}_{1}=x_{1}+2 x_{2}  \tag{13.15}\\
& \dot{x}_{2}=-2 x_{1}+x_{2} \tag{13.16}
\end{align*}
$$

Using standard state-space notation:

$$
\dot{\mathbf{x}}=\mathbf{A} \mathbf{x}
$$

The Jacobian matrix is:

$$
A=\left[\begin{array}{cc}
1 & 2 \\
-2 & 1
\end{array}\right]
$$

with eigenvalues $1 \pm 2 j$. This system is unstable because the real portion of the complex eigenvalues is positive.

The phase-plane plot is shown in Figure 13.5.


FIGURE 13.5 Example 13.4, unstable focus (spial source).

Another type of linear system behavior occurs when the eigenvalues have a zero real portion. That is, the eigenvalues are on the real axis. This type of system leads to closed curves in the phase-plane, and is known as center behavior. Example 13.5 illustrates center behavior.

## EXAMPLE 13.5 Center

Consider the following system of equations:

$$
\begin{align*}
& \dot{x}_{1}=-x_{1}-x_{2}  \tag{13.17}\\
& \dot{x}_{2}=4 x_{1}+x_{2} \tag{13.18}
\end{align*}
$$

The Jacobian matrix is:

$$
A=\left[\begin{array}{rr}
-1 & -1 \\
4 & 1
\end{array}\right]
$$

and the eigenvalues are $0 \pm 1.7321 \mathrm{j}$. Since the real part of the eigenvalues is zero, there is a periodic solution (sine and cosine), resulting in a phase-plane plot where the equilibrium point is a ceuter, as shown in Figure 13.6.


FIGURE 13.6 Example 13.5, eigenvalues with zero real portion are centers.

Examples 13.1 to 13.5 we have provided an introduction to linear system phase-plane behavior. We noted the important role of eigenvectors and eigenvalues, and how these relate to the concept of a separatrix. Section 13.3 provides a generalization of these examples.

### 13.3 GENERALIZATION OF PHASE-PLANE BEHAVIOR

We wish now to generalize our results for second-order linear systems of the form:

$$
\begin{equation*}
\dot{\mathbf{x}}=\mathbf{A} \mathbf{x} \tag{13.19}
\end{equation*}
$$

where the Jacobian matrix is:

$$
A=\left[\begin{array}{ll}
a_{11} & a_{12}  \tag{13.20}\\
a_{21} & a_{22}
\end{array}\right]
$$

Recall that the eigenvalues are found by solving $\operatorname{det}(\lambda I-A)=0$ :

$$
\begin{equation*}
\operatorname{det}(\lambda I-A)=\left(\lambda-a_{11}\right)\left(\lambda-a_{22}\right)-a_{12} a_{21}=0 \tag{13.21}
\end{equation*}
$$

which can be written as:

$$
\begin{equation*}
\operatorname{det}(\lambda I-A)=\lambda^{2}-\operatorname{tr}(A) \lambda+\operatorname{det}(A)=0 \tag{13.22}
\end{equation*}
$$

The quadratic formula can be used to find the eigenvalues:

$$
\begin{equation*}
\lambda=\frac{\operatorname{tr}(A) \pm \sqrt{(\operatorname{tr}(\Lambda))^{2}-4 \operatorname{det}(A)}}{2} \tag{13.23}
\end{equation*}
$$

or, expressing each eigenvalue separatcly,

$$
\lambda_{1}=\frac{\operatorname{tr}(A)-\sqrt{(\operatorname{tr}(A))^{2}-4 \operatorname{det}(A)}}{2}
$$

and,

$$
\lambda_{2}=\frac{\operatorname{tr}(A)+\sqrt{(\operatorname{tr}(A))^{2}-4 \operatorname{del}(A)}}{2}
$$

We notice that at least one cigenvalue will be negative if $t(A)<0$. We also notice that the eigenvalues will be complex if $4 \operatorname{det}(A)>\operatorname{tr}(A)^{2}$. Remember that the different behaviors resulting from $\lambda_{1}$ and $\lambda_{2}$ are:

| Sinks (stable nodes): | $\operatorname{Re}\left(\lambda_{1}\right)<0$ and $\operatorname{Re}\left(\lambda_{2}\right)<0$ |
| :--- | :--- |
| Saddles (unstable): | $\operatorname{Re}\left(\lambda_{1}\right)<0$ and $\operatorname{Re}\left(\lambda_{2}\right)>0$ |
| Sources (unstable nodes): | $\operatorname{Re}\left(\lambda_{1}\right)>0$ and $\operatorname{Re}\left(\lambda_{2}\right)>0$ |
| Spirals: | $\lambda_{1}$ and $\lambda_{2}$ are complex complex conjugates. If $\operatorname{Re}\left(\lambda_{1}\right)$ |
|  | $<0$ then stable, if $\operatorname{Re}\left(\lambda_{1}\right)>0$ then unstable. |

We can then use Figure 13.7 to find the phase-plane behavior for second-order linear ordinary differential equations as a function of the trace and determinant of A. In Figure


FGGRE 13.7 Dynamic behavior diagran for second-order linear systems.
The $x$-axis is $\operatorname{tr}(A)$ and the $y$-axis is det $(A)$.
13.7, the $x$-axis is the trace of $A$ and the $y$-axis is the determinant of $A$. For example, consider Example 13.1, where

$$
\begin{aligned}
A & =\left[\begin{array}{rr}
-1 & 0 \\
0 & -4
\end{array}\right] \\
\operatorname{tr}(A) & =-5 \\
\operatorname{det}(A) & =4
\end{aligned}
$$

The point $(-5,4)$ lies in the second quadrant in the node sink sector, as expected, since the two real eigenvalues are negative (indicting stable node behavior).

Figure 13.8 shows the phase-plane behavior as a function of the eigenvalue locations in the complex plane. For example, two negative eigenvalues lead to stable node behavior.

### 13.3.1 Slope Marks for Vector Fields

A qualitative assessment of the phase-plane behavior can be obtained by ploting the slope marks for the vector field. Consider a general linear 2-state system

## Eigenvalues Phase-Plane Plot



FIGURE 13.8 Phasc-plane behavior as a function of Eigenvalue location.

Eigenvalues
Phase-Plane Plot


FIGURE 13.8 Contimued

$$
\begin{align*}
& \dot{x}_{1}=a_{11} x_{1}+a_{12} x_{2}  \tag{13.24}\\
& \dot{x}_{2}=a_{21} x_{1}+a_{22} x_{2} \tag{13.25}
\end{align*}
$$

We can divide (13.25) by (13.24) to find how $x_{2}$ changes with respect to $x_{1}$ :

$$
\begin{equation*}
\frac{d x_{2}}{d x_{1}}=\frac{a_{21} x_{1}+a_{22} x_{2}}{a_{11} x_{1}+a_{12} x_{2}} \tag{13.26}
\end{equation*}
$$

and we can plot "slope marks" for values of $x_{1}$ and $x_{2}$ to determine an idea of how the phase plane will look. Let us revisit Example 13.3. The slope marks can be calculated from (13.27):

$$
\begin{equation*}
\frac{d x_{2}}{d x_{1}}=\frac{2 x_{1}-x_{2}}{2 x_{1}+x_{2}} \tag{13.27}
\end{equation*}
$$

Figure 13.9 shows the slope marks for Example 13.3. These are generated by forming a grid of points in the plane, and finding the slope associated with each point; short line segments with the slope calculated are then plotted for each point. Notice that one can use the slope marks to help sketch state variable trajectories, as shown in Figure 13.10. Saddle point behavior found in Example 13.3 is clearly shown in Figure 13.10.

### 13.3.2 Additional Discussion

Phase-plane analysis can be used to analyze autonomous systems with two state variables. Notice that state variable trajectories cannot "cross" in the plane, as illustrated by the following reasoning. Think of any point of a trajectory as being an initial condition. The model, when integrated from that initial condition, must have a single trajectory. If two trajectories crossed, that would be the equivalent of saying that a single initial condition could have two different trajectories. If a system was non-autonomous (for example, if there was a forcing function that was a function of time) then state variable trajectories could cross, because a model with the same initial conditions but a different forcing function would have different trajectories.

An autonomous (unforced) system with $n$ state variables cannot have trajectories that cross in $n$-space, but may have trajectories that cross in less than n-space. For exam-


FIGURE 13.9 Slope marks for the vector field of Example 13.3.


FIGURE 13.10 Slope marks with trajectories for Example 13.3.
ple, a third-order autonomous system cannot have trajectories that cross in 3 -space, but the trajectories may cross when placed in a two-dimensional plane.

### 13.4 NONLINEAR SYSTEMS

In the previous sections we discovered the types of phase-plane behavior that could be observed in linear systems. In this chapter we will find that nonlinear systems will often have the same general phase-plane behavior as the model linearized about the equilibrium (steady-state) point, when the system is close to that particular equilibrium point.

In this section we study two examples. Example 13.6 is based on a simple bilinear model, while Example 13.7 is a classical bioreactor model.

## EXAMPLE 13.6 Nonlinear (Bilinear) System

Consider the following system:

$$
\begin{align*}
& \frac{d z_{1}}{d t}=z_{2}\left(z_{1}+1\right)  \tag{13.28}\\
& \frac{d z_{2}}{d t}=z_{1}\left(z_{2}+3\right) \tag{13.29}
\end{align*}
$$

which has two steady-state (equilibrium) solutions:

| Equilibrium 1: trivial | $z_{1 s}=0$ | $z_{2 s}=0$ |
| :--- | :--- | :--- |
| Equilibrium 2: nontrivial | $z_{1 s}=-1$ | $z_{2 s}=-3$ |

Linearizing (13.28) and (13.29), we find the following Jacobian matix:

$$
A=\left[\begin{array}{cc}
z_{2 x} & z_{1,}+1 \\
z_{2 s}+3 & z_{1,5}
\end{array}\right]
$$

In the following, we analyze the stability of each equilibrium point.

## Equilibrium 1 (Trivial)

The Jacobian matrix is: $\quad A=\left[\begin{array}{ll}0 & 1 \\ 3 & 0\end{array}\right]$
and the eigenvalues are: $\quad \lambda_{1}=-\sqrt{3} \quad \lambda_{2}=\sqrt{3}$
We know from linear system analysis that equilibrian point one is a sadde point, since one eigenvalue is stable and the other is anstable.

The stable eigenvector is: $\quad \xi_{1}=\left[\begin{array}{c}-0.5 \\ 0.866\end{array}\right]$

The unstable eigenvector is: $\xi_{2}=\left[\left.\begin{array}{c}0.5 \\ 0.806\end{array} \right\rvert\,\right.$
The phase-plane of the lineatized model around equilibritm point one is a sadde, as shown in Figure 13.11. The lincarized model is


FIGURE 13.11 Phase-plame of the Example 13.6 model linearized aromd trivial equibrium point. This point is a sadtle point.
where $\mathrm{x}=\boldsymbol{z}-\boldsymbol{z}_{\mathrm{s}}$

## Equilibrium 2 (Nontrivial)

The Jacobian matrix is:

$$
\dot{\mathbf{x}}=\left|\begin{array}{cc}
0 & 1 \\
3 & 0
\end{array}\right| \mathbf{x}
$$

$$
A=\left[\begin{array}{cc}
-3 & 0 \\
0 & -1
\end{array}\right]
$$

the eigenvalues are:

$$
\lambda_{1}=-3 \quad \lambda_{2}=-1
$$

So, we know from linear system analysis that equilibriun point two is a stable mode, since both cigenvalues are stable.
The "fast" stable eigenvector is $\quad \xi_{1}=\left[\begin{array}{l}1 \\ 0\end{array}\right]$
The "slow" stable eigenvector is $\quad \xi_{2}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$
The phase-plane of the linearized model around equilbrium point two is a stable node, as shown in Figure 13.12.


FIGURE 13.12 Phase-plane of the Example 13.6 model linearized around triviat equibrium poin. This point is a stable node.

The phase-plane diagram of the nonlinear model is shown in Figure 13.13. Notice how the lincarized models capture the behavior of the nonlincar model when close to one of the equilbrium points. Notice, however, that initial conditions inside the "right" sadde "blow up," while initial conditions inside the left saddle are attracted to the stable point. Slope-fiefd marks are shown in Figure 13.14.


IIGURE 13.13 Phase-plane of Example 13.6. Trajectories (except those of the right side of the sadde) leave the unstable point and are "attracted" to the stable point.


FIGURE 13.14 Slope-field marks and some trajectones for Example 13.6.

## EXAMPLE 13.7 Bioreactor with Monod Kineties

Consider a model for a bioreactor with Monod kinetics (see Module 8):

$$
\begin{gather*}
\frac{d x_{1}}{d t}=(\mu-D) x_{1}  \tag{13.30}\\
\left.\frac{d x_{2}}{d t}=\left(s_{f}-x_{2}\right) D\right)-\frac{\mu x_{1}}{\gamma}  \tag{13.31}\\
\mu=\frac{\mu_{m a x} x_{2}}{k_{m}+x_{2}} \tag{13.32}
\end{gather*}
$$

where:

$$
\begin{array}{rlrl}
\mu_{\max } & =0.53 & k_{m}=0.12 \\
Y & =0.4 & & s_{f}=4.0
\end{array}
$$

$x_{1}$ is the biomass concentration and $x_{2}$ is the substrate concentration. There are two steady-state (equilibrium) solutions for this set of parameters.

Equilibrium 1: trivial
Equilibrium 2: nontrivial

$$
x_{1:}=0
$$

$$
x_{2 s}=4.0
$$

$$
x_{1 s}=1.4523
$$

$$
x_{2 s}=0.3692
$$

Lineariving (13.30) and (13.31) we find the following Jacobian matrix:

$$
\mathbf{A}=\left[\begin{array}{cc}
\mu_{s}-D_{s} & x_{1 s} \mu_{s}^{\prime} \\
-\frac{\mu_{s}}{Y} & -D_{s}-\frac{\mu_{s}^{\prime} x_{1 s}}{Y}
\end{array}\right]
$$

where we have defined $\mu^{\prime}=\frac{\partial \mu_{1}}{\partial x_{2}}=\ldots \mu_{2 s}\left(k_{m}+x_{2 v}\right)$

## Equilibrium 1 (Trivial)

The Jacobian matrix is: $\quad A=\left[\begin{array}{cc}0.114563 & 0 \\ -1.286408 & -0.4\end{array}\right]$
with eigenvalues of: $\quad \lambda_{1}=0.114563$ and $\quad \lambda_{2}=-0.4$
indicating that the steady-state is unstable (it is a saddle point).
The unstable cigenvector is: $\quad \xi_{i}=\left[\begin{array}{r}0.3714 \\ -0.9285\end{array}\right]$
The stable eigenvector is: $\quad \xi_{2}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$
This steady-state is known as the "wash-out" steady-state, because no biomass is produced and the substrate concentration in the reactor is equal to the feed substrate concentration.

## Equilibrium 2 (Nontrivial)

The Jacobian matrix is:

$$
A=\left[\begin{array}{rr}
0 & 3.215929 \\
-1 & -8.439832
\end{array}\right]
$$

with eigenvalues of

$$
\lambda_{1}=-0.4 \quad \text { and } \quad \lambda_{2}=-8.0398
$$

indicating that the steady-state is a stable node.

The "slow" stable eigenvector is

$$
\xi_{1}=\left[\begin{array}{r}
0.9924 \\
-0.11234
\end{array}\right]
$$

The "fast" stable eigenvector is

$$
\xi_{2}=\left[\begin{array}{l}
0.3714 \\
0.9285
\end{array}\right]
$$

The phase-plane plot of Figure 13.15 shows that the trajectories leave unstable point I $(0,4)$ and go to stable point $2(1.4523,0.3692)$. More detail of the phase-plane around the unstable point is shown in Figure 13.16, while Figure 13.17 shows more detail around the stable point.


FIGURE 13.15 Phase plane for bioreactor with Monod kinetics. x is biomass concentration and $s$ is substrate concentration.


FIGURE 13.16 Phase-plane behavior near the unstable point (0,4) (Equilibrium 1).


FIGURE 13.17 Phase-plane behavior near the stable point (1.4523, 0.3692)
(Equilibrium 2).


FIGURE 13.18 Example of center behavior.

In the previous examples the system trajectories "left" an unstable point and were "attracted" to a stable point. Another type of behavior that can occur is limit cycle or periodic behavior. This is illustrated in the following section.

### 13.4.1 Limit Cycle Behavior

In Section 13.2 we noticed that linear systems that had eigenvalues with zero real portion formed centers in the phase plane. The phase-plane trajectories of the systems with centers depended on the initial condition values. An example is shown in Figure 13.18. A somewhat related behavior that can occur in nonlinear systems is known as limit cycle behavior, as shown in Figure 13.19.

The major difference in center (Figure 13.18) and limit cycle (Figure 13.19) behavior is that limit cycles are isolated closed orbits. By isolated, we mean that an inital per-


FIGURE 13.19 Example of limit cycle behavior.
turbation from the closed cycle eventually returns to the closed cycle. Contrast that with center behavior, where a perturbation leads to a different closed cycle.

Limit cycle behavior will be discussed in more detail in Chapter 16.

## SUMMARY

As noted earlier, phase-plane analysis is a useful tool for observing the behavior of nonlineat systems. We have spent time analyzing autonomous linear systems, because the nonlinear systems will behave like a linear system, in the vicinity of the equibibrium point (where the linear approximation is most valid). A qualitative feel for phase-plane behavior can be obtained by plotting slope marks.

Notice that we have shown examples of nonlinear systems that have multiple equilibrium points (steady-state solutions). Phase-plane analysis can be used to determine regions of initial conditions where a system may converge to one (stable) equilbrium point and regions where the initial conditions may converge to another (stable) equilibrium point.

By sketching the linear behavior around a particular equilibrium point and by using slope marks, we can qualitatively sketch the phase-plane behavior of a given nonlinear system.

Clearly the phase-plane approach is limited to systems with two state variables. Analogous procedures can be used to develop phase-space plots in three dimensions for three-state systems. Linearization and analysis of the locally linear behavior in terms of eigenvalues and eigenvectors can still be used for higher-order systems, but the phase behavior cannot be viewed for these higher-order systems.

## FURTHER READING

Strogatz, S.H. (1994). Nonlinear Dynamics and Chaos. Reading, MA: Addison Wesley.

## STUDENT EXERCISES

## Linear Problems

For the following linear systems, use Figure 13.7 to determine the phase-plane behavior. Also, calculate the eigenvalues and use Figure 13.8 to verify your results. Develop your own phase-plane diagrams for any situations not covered in Figures 13.7 and 13.8.

1. $\dot{\mathbf{x}}=\left|\begin{array}{rr}-1 & 0 \\ 1 & -2\end{array}\right| \mathbf{x}$
2. $\dot{\mathbf{x}}=\left[\begin{array}{rr}-1 & 3 \\ 2 & -2\end{array}\right] \mathbf{x}$
3. $\dot{\mathbf{x}}=\left[\begin{array}{ll}1 & 3 \\ 2 & 2\end{array}\right] \mathbf{x}$
4. $\dot{\mathbf{x}}=\left[\begin{array}{ll}1 & 0 \\ 1 & 2\end{array}\right] \mathbf{x}$
5. $\quad \dot{\mathbf{x}}=\left[\begin{array}{rr}1 & -2 \\ 1 & 2\end{array}\right] \mathbf{x}$
6. $\dot{\mathbf{x}}=\left[\begin{array}{rr}-1 & -2 \\ 1 & -2\end{array}\right] \mathbf{x}$
7. Compare $\dot{\mathbf{x}}=\left[\begin{array}{rc}-1 & -0.25 \\ 1 & -2\end{array}\right] \mathbf{x}$ with $\dot{\mathbf{x}}=\left[\begin{array}{cc}1 & -0.25 \\ 1 & 2\end{array}\right] \mathbf{x}$
8. $\quad \dot{x}=\left[\begin{array}{rr}-1 & -0.5 \\ 2 & 1\end{array}\right] \mathbf{x}$
9. $\quad \dot{\mathbf{x}}=\left[\begin{array}{rr}-1 & 1 \\ 2 & -2\end{array}\right] \mathbf{x}$
10. $\dot{\mathbf{x}}=\left[\begin{array}{rrr}-1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3\end{array}\right] \mathbf{x}$
11. $\dot{\mathbf{x}}=\left[\begin{array}{rrr}-1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3\end{array}\right] \mathbf{x}$
12. A process engineer has linearized a nonlinear process model to obtain the following state-space model and given it to your boss. Your boss has forgoten everything he learned on dynamic systems and has asked you to study this model using linear system analysis techniques.

$$
\dot{\mathbf{x}}=\mathbf{A} \mathbf{x}
$$

where:

$$
\mathbf{A}=\left[\begin{array}{cc}
0 & -1.0 \\
1.0 & 0.0
\end{array}\right]
$$

with initial conditions $x_{1}(0)=0.5$ and $x_{2}(0)=-0.25$.
a. What are the eigenvalues of the A matrix? Use both MATLAB and your own analytical solution.
b. Show a phase-plane plot, placing $x_{1}$ on the $x$-axis and $x_{2}$ on the $y$-axis.
13. Consider a process with a state-space $A$ (Jacobian) matrix that has the following eigenvalues and eigenvectors. Draw the phase-plane plot, clearly showing the direction of the trajectories. The eigenvalues are:

$$
\lambda_{1}=-1 \quad \lambda_{2}=1
$$

and the eigenvectors are:

$$
\xi_{1}=\left[\begin{array}{c}
\sqrt{2} \\
2 \\
\sqrt{2} \\
2
\end{array}\right] \quad \xi_{2}=\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

14. An interesting example of phase-plane behavior is presented in the book by Strogatz (1994). He develops a simple model for love affairs, using Romeo and Juliet to illustrate the concepts. Consider the case where Romeo is in love with Juliet, but Juliet is fickle. The more that Romeo loves her, the more that Juliet resists his love. When Romeo becomes discouraged and backs off, Juliet becomes more attracted to him. Let:

$$
\begin{aligned}
& x_{1}=\text { Romeo's love/hate for Juliet } \\
& x_{2}=\text { Juliet's love/hate for Romeo }
\end{aligned}
$$

where positive state variable values indicate love and negative values indicate hate. The model for this relationship is:

$$
\begin{aligned}
& \frac{d x_{1}}{d t}=a x_{2} \\
& \frac{d x_{2}}{d t}=-b x_{1}
\end{aligned}
$$

where $a$ and $b$ are positive parameters. Show that this model has center behavior and discuss the meaning from a romance perspective.
15. Consider a more general formulation of the Romeoluliet problem in 14 above. In this case, let:

$$
\begin{aligned}
& \frac{d x_{1}}{d t}=a_{11} x_{1}+a_{12} x_{2} \\
& \frac{d x_{2}}{d t}=a_{21} x_{1}+a_{22} x_{2}
\end{aligned}
$$

where the parameters $a_{i j}$ can be either positive or negative. The choice of signs specifies the romantic "styles." For each of the following cases (parameters $a$ and $b$ are positive), determine the phase-plane behavior. Interpret the meaning of the results in terms of romantic behavior.
a. $\frac{d x_{1}}{d t}=a x_{1}+b x_{2}$

$$
\frac{d x_{2}}{d t}=b x_{1}+a x_{2}
$$

b. $\frac{d x_{1}}{d t}=-a x_{1}+b x_{2}$
$\frac{d x_{2}}{d t}=b x_{1}-a x_{2}$
c. $\frac{d x_{1}}{d t}=a x_{1}+b x_{2}$

$$
\frac{d x_{2}}{d t}=-b x_{1}-a x_{2}
$$

## Nonlinear Problems

16. As a chemical engineer in the pharmaceutical industry you are responsible for a process that uses a bacteria to produce an antibiotic. The reactor has been contaminated with a protozoan that consumes the bacteria. Assume that predator-prey equations are used to model the system ( $x_{1}=$ bacteria (prey), $x_{2}=$ protozoa (predator)). The time unit is days.

$$
\begin{aligned}
& \frac{d x_{1}}{d t}=\alpha x_{1}-\gamma x_{1} x_{2} \\
& \frac{d x_{2}}{d t}=\varepsilon \gamma x_{1} x_{2}-\beta x_{2}
\end{aligned}
$$

a. Show that the nontrivial steady-state values are:

$$
x_{1 s}=\frac{\beta}{s \gamma} \quad x_{2 s}=\frac{\alpha}{\gamma}
$$

b. Use the scaled variables, $y_{1}$ and $y_{2}$,

$$
y_{1}=\frac{x_{1}}{x_{1 s}} \quad y_{2}=\frac{x_{2}}{x_{2 s}}
$$

to find the scaled modeling equations:

$$
\begin{aligned}
\frac{d y_{1}}{d t} & =\alpha\left(1-y_{2}\right) y_{1} \\
\frac{d y_{2}}{d t} & =-\beta\left(1-y_{1}\right) y_{2}
\end{aligned}
$$

c. Find the eigenvalues of the Jacobian matrix for scale equations, evaluated at $y_{1} s$ and $y_{2} s$. Realize that $y_{1} s$ and $y_{2} s$ are 1.0 by definition. Find the eigenvalues in terms of $\alpha$ and $\beta$.
d. The parameters are $\alpha=\beta=1.0$ and the initial conditions are $y_{1}(0)=1.5$ and $y_{2}(0)=0.75$.
i. Plot the transient response of $y_{1}$ and $y_{2}$ as a function of time (plot these curves on the same graph using MATLAB). Using your choice of integration methods, simulate to at least $t=20$.
ii. Show a phase-plane plot, placing $y_{1}$ on the $x$-axis and $y_{2}$ on the $y$-axis.
iii. What is the "peak-to-peak" time for the bacteria? By how much time does the protozoa "lag" the bacteria?
e. Now consider the trivial steady-state $\left(x_{1 s}=x_{2 s}=0\right)$, Is it stable? Perform simulations when $x_{1}(0) \neq 0$ and $x_{2}(0) \neq 0$. What do you find?
f. What if $x_{1}(0) \neq 0$ and $x_{2}(0)=0$ ?
g. What if $x_{1}(0)=0$ and $x_{2}(0) \neq 0$ ?
17. Consider the bioreactor model used in Example 13.7 with substrate inhibition rather than Monod kinetics (see Module 8 for more detail)

$$
\begin{aligned}
\frac{d x_{1}}{d t} & =(\mu-D) x_{1} \\
\frac{d x_{2}}{d t} & =\left(s_{j}-x_{2}\right) D-\frac{\mu x_{1}}{Y} \\
\mu & =\frac{\mu_{\max } x_{2}}{k_{m}+x_{2}+k_{1} x_{2}^{2}}
\end{aligned}
$$

where:

$$
\begin{aligned}
\mu_{\max } & =0.53 & k_{m}=0.12 \\
Y & =0.4 & s_{f}=4.0 \\
k_{1} & =0.4545 &
\end{aligned}
$$

and $x_{1}$ is the biomass concentration and $x_{2}$ is the substrate concentration.
Assume that the steady-state dilution rate is $D_{s}=0.3$.
a. Find the steady-state (cquilibrium) solutions (Hint: There are three).
b. Analyze the stability of each steady-state. Find the Jacobian, the eigenvalues, and the eigenvectors at each steady-state.
c. Construct a phase plane plot. What do you observe about the unstable steadystate?
d. What would you do if it was desirable to operate the reactor at an unstable steady-state?
18. Perform some time domain plots related to the phase-plane plots for Example 13.7. Discuss how these plots relate to the phase-plane results.
19. A chemical reactor that bas a single second-order reaction and has an outlet flowrate that is a lincar function of height has the following model where the outlet flowrate is lincarly related to the volume of liquid in the reactor $(F=\beta V)$.

$$
\begin{aligned}
\frac{d C}{d t} & =\frac{F_{i n}}{V}\left(C_{i n}-C\right)-k C^{2} \\
\frac{d V}{d t} & =F_{i n}-\beta V
\end{aligned}
$$

The parameters, variables and their steady-state values are shown below:
$F_{i n}=$ inlet flowate ( 1 liter/min)
$C_{i n}=$ inlet concentration ( 1 gmol/liter)
$C=$ reactor concentration ( $0.5 \mathrm{gmol} / \mathrm{liter}$ )
$V=$ reacto volume ( 1 liter)
$k=$ reaction rate constant ( 2 liter/(gmol min))
$\beta=1 \mathrm{~min}^{-1}$
Perform a phase-plane analysis and discuss your results.
20. Consider two interacting tanks in series, with outle flowrates that are a function of the square root of tank height. The flow from tank 1 is a function of $\sqrt{h_{1}-h_{2}}$, while the flowrate out of tank 2 is a function of $V h_{2}$


The following modeling equations describe this system:

$$
\left[\begin{array}{c}
\frac{d h_{1}}{d t} \\
\frac{d h_{2}}{d t}
\end{array}\right]=\left[\begin{array}{l}
f_{1}\left(h_{1}, h_{2}, F\right) \\
f_{2}\left(h_{1}, h_{2}, F\right)
\end{array}\right]=\left[\begin{array}{c}
F \\
A_{1}-\beta_{1} \\
A_{1} \\
\sqrt{h_{1}}-h_{2} \\
\frac{\beta_{1}}{A_{2}} \sqrt{h_{1}-h_{2}}-\frac{\beta_{2}}{A_{2}} \sqrt{h_{2}}
\end{array}\right]
$$

For the following parameter values

$$
\beta_{1}=2.5 \frac{\mathrm{ft}^{2.5}}{\min } \quad \beta_{2}=\frac{5 \mathrm{ft}^{2.5}}{\sqrt{6} \frac{\min }{}} \Lambda_{1}=5 \mathrm{ft}^{2} \quad A_{2}=10 \mathrm{ft}^{2}
$$

and the input

$$
F=5 \frac{\mathrm{ft}^{3}}{\mathrm{~min}}
$$

The steady-state height values are

$$
h_{1 s}=10 \quad h_{2 s}=6
$$

Perform a phase-plane analysis and discuss your results.

# INTRODUCTION TO NONLINEAR DYNAMICS: A CASE STUDY OF THE QUADRATIC MAP 

### 14.5 Bifurcation and Orbit Diagrams

14.6 Stability of Fixed-Point Solutions
14.7 Cascade of Period-Doublings
14.8 Further Comments on Chaotic Behavior

### 14.1 BACKGROUND

Many engineers and scientists have assumed (at least, until roughly twenty years ago) that simple models have simple solutions and simple behavior, and that this behavior is predictable. Indeed, the main objective for developing a model is usually to be able to predict behavior or to match observed behavior (measured data). During the past thirty years, a number of scientists and engineers have discovered simple models where the shor-term behavior is predictable, but sensitivity to intial conditions make the long-term prediction impossible. By initial conditions, we mean the value of the variables at the beginning of the integration in time. An example is the simple weather prediction model of Lorenz. (1963), which is a system of three nonlinear ordinary differential equations; the Lorenz. model is covered in more detail in Chapter 17. Another example is the population growth model used by May (1976), which is a single nonlinear discrete time equation. This population model is the topic of this chapter.

The commonly accepted term for the dynamic hehavior of a system that exhibits sensitivity to initial conditions is chaos. Terms for the branch of mathematics related to chaos include nonlinear dynamics, dynamical systems theory, or nonlinear science. New chaos books, written for a general audience, appear frequently; some of the more interesting ones are referenced at the end of this chapter.

This chapter will not make you an expert on nonlinear dynamics, but it will help you understand what is meant by sensitivity to initial conditions and practical limits to long-term predictability.

### 14.2 A SIMPLE POPULATION GROWTH MODEL

Assume that the population of a species during one time period is a function of the previous time period. Perhaps we are interested in the number of bacteria cells that are growing in a petri dish. or maybe we are concerned about the population of the United States. In either case, the mathematical model is:

$$
\begin{equation*}
n_{k+1}=n_{k}+b_{k}-d_{k} \tag{14.1}
\end{equation*}
$$

where | $n_{k}$ | $=$ |
| ---: | :--- |
| $b_{k}$ | $=$ |
| $b_{k}$ | population at the beginning of time period $k$ |
| $d_{k}$ | $=$ number of births during time period $k$ |

Now assume that the number of births and deaths during time period $k$ is proportional to the population at the beginning of time period $k$.

$$
\begin{align*}
& b_{k}=\alpha_{b} n_{k}  \tag{14.2}\\
& d_{k}=\alpha_{d} n_{k} \tag{14.3}
\end{align*}
$$

where $\alpha_{b}$ and $\alpha_{d}$ are birth and death constants. Then:

$$
\begin{equation*}
n_{k+1}=n_{k}+\alpha_{b} n_{k} \cdots \alpha_{d} n_{k} \tag{14.4}
\end{equation*}
$$

which we can write as:

$$
\begin{align*}
& n_{k+1}=n_{k}+\left(\alpha_{b}--\alpha_{d}\right) n_{k}  \tag{14.5}\\
& n_{k+1}=(1+r) n_{k} \tag{14.6}
\end{align*}
$$

or,
where $r=\alpha_{b}-\alpha_{d}$ Eqn. (14.6) can be simply written as:

$$
\begin{equation*}
n_{k+1}=\alpha n_{k} \tag{14.7}
\end{equation*}
$$

where $\alpha=1+r=1+\alpha_{b}-\alpha_{d}$ (obviously, $\alpha>0$ for a physical system)
The analytical solution to (14.7) is:

$$
\begin{equation*}
n_{k}=\alpha^{k} n_{0} \tag{14.8}
\end{equation*}
$$

where $n_{0}$ is the initial condition.
From inspection of (14.8) we observe that
if $\alpha<1$ The population decreases during each time period (converging to 0 ).
if $\alpha>1$ The population increases during each time period $(\rightarrow \infty)$.
if $\alpha=1$ The population remains constant during each time period.
These results are also shown in Figure 14.1


FIGURE 14.1 Simple population growth model.

These results are casily rationalized, since births<deaths, births>deaths, and births = deaths for the three cases. The result for $\alpha>1$ is consistent with Malthus, who in the nineteenth century predicted an exponential population growth.

The result that the population increases to $\infty$ for $\alpha>1$ is a bit unrealistic. In practice, the amount of natural resources available will limit the total poputation (for the bacteria case, the amount of nutrients or the size of the Petri dish will limit the maximum number of bacteria that can be grown). In the next example, we show a simple model that "constrains" the maximum population.

### 14.3 A MORE REALISTIC POPULATION MODEL

A common model that has been used to predict population growth is known as the logistic equation or the quadratic map (May, 1976).

$$
\begin{equation*}
x_{k+1}=\alpha x_{k}\left(1-x_{k}\right) \tag{14.9}
\end{equation*}
$$

Here, $x_{k}$ represents a scaled population variable (see student exercise 3 ).
Note the similarity of (14.9) with the numerical methods presented in Chapter 3:

$$
\begin{equation*}
x_{k+1}=g\left(x_{k}\right) \tag{14.10}
\end{equation*}
$$

Recall that direct substitution is sometimes used to solve a nonlinear algebraic equation. The next guess (iteration $k+1$ ) for the variable that is being solved for $(x)$ is a lunction of the current guess (iteration $k$ ). Equation (14.9) shows how the population changes from time period to time period-that is, it is a discrete dynamic equation. Since (14.9) is the same form as (14.10), we will learn a lot about the quadratic map from analysis of the direct substitution technique and vice versa. You will also note that many numerical integration techniques (Euler, Runge-Kutta) have the form of (14.10).

Since (14.9) is a discrete dynamic equation, we can determine the steady-state behavior by finding the solution as $k \rightarrow \infty$. Writing (14.9) in a more explicit form,

$$
\begin{equation*}
x_{k-11}=\alpha x_{k}-\alpha x_{k}^{2} \tag{14.11}
\end{equation*}
$$

as we approach a steady-state (fixed-point) solution, $x_{k+1}=x_{k}$, so we can write:

$$
\begin{equation*}
x_{s}=\alpha x_{s}-\alpha x_{s}^{2} \tag{14.12}
\end{equation*}
$$

which can be written:

$$
\begin{equation*}
\alpha x_{s}^{2}-(\alpha-1) x_{s}=0 \tag{14.13}
\end{equation*}
$$

We can use the quadratic formula to find the steady-state (fixed-point) solutions:

$$
\begin{equation*}
x_{s}=0 \text { and } \frac{\alpha-1}{\alpha} \tag{14.14}
\end{equation*}
$$

It is easy to see from (14.9) that if the initial population is zero, it will remain at zero. For a non-zero initial condition, one would expect convergence (steady-state) of the population to $(\alpha-1) / \alpha$. We will use a case study approach to show that the actual long-term

TABLE 14.1 Parameters and Nonzero Solutions for Four Cases

| Case | $\alpha$ | $x_{s}$ |
| :---: | :---: | :---: |
| 1 | 2.95 | 0.6610 |
| 2 | 3.20 | 0.6875 |
| 3 | 3.50 | 0.7143 |
| 4 | 3.75 | 0.7333 |

(steady-state) behavior can be quite complex. Table 14.1 shows the $\alpha$ parameter and the non-zero steady-state that is expected from (14.14).

### 14.3.1 Transient Response Results for the Quadratic (Logistic) Map

Each case presented in Table 14.1 has distinctly different dynamic behavior. As shown in the following sections, case 1 illustrates asymptotically stable behavior, cases 2 and 3 illustrate periodic behavior, and case 4 illustrates chaotic behavior.

## ASYMPTOTICALLY STABLE BEHAVIOR

Let $x_{0}$ represent the initial condition (the value of the population at the initial time) and $x_{k}$ represent the population value at time step $k$. For case I we find the following values, using the relationship $x_{k+1}=2.95 x_{k}\left(1-x_{k}\right)$ :

| Step $k$ | $x_{k}$ | $x_{k+1}$ |
| :---: | :---: | :---: |
| 0 | 0.1 | 0.2655 |
| 1 | 0.2655 | 0.5753 |
| 2 | 0.5753 | 0.7208 |
| 3 | 0.7208 | 0.5937 |
| 4 | 0.5937 | 0.7116 |
| 5 | 0.7116 | 0.6054 |
| $\infty$ | 0.6610 | 0.6610 |

The transient response for case 1 is plotted in Figure 14.2 for an initial condition of 0.1. Notice that the response converges to the predicted steady-state of 0.6610 . This type of response for continuous models is usually called asymptotically stable behavior since the output converges to the steady-state (fixed-point) solution.

## PERIODIC BEHAVIOR

The transient response curve for case 2 is shown in Figure 14.3. The curve oscillates between 0.513 and 0.800 , while the predicted result (equation 14.14 and Table 14.1) is


FGURE 14.2 Transient population response, case 1 ; converges to single steady-state.


FIGURE 14.3 Transient population response, case 2; oscillates between two values (period-2 behavior).
0.6875 . This type of response is known as period-2 behavior. In case 3 the transient response oscillates between 0.383, 0.827, 0.501, and 0.875 as shown in Figure 14.4. This is known as period-4 behavior, since the system returns to the same state value every fourth time step.

## CHAOTIC BEHAVIOR

For Case 4, the transient response never settles to a consistent set of values, as shown in Figure 14.5; rather the values appear to be somewhat "random" although a deterministic equation has been used to solve the problem. Figure 14.6 shows that a slight change in initial condition from 0.100 to 0.101 leads to a significantly different point-to-point re-sponse-this is known as sensitivity to initial conditions and is characteristic of chaotic systems.

## WHERE WE ARE HEADING

At this point, you are probably wondering how to predict the type of behavior that the quadratic map is going to have. Changes in the $\alpha$ parameter have led to many different types of behavior. The purpose of Section 14.4 is to show how to predict the type of behavior that will be observed using cobweb diagrams. Section 14.5 will then introduce bifurcation plots, which reduce the long-term results from many transient plots to a single plot. Section 14.6 introduces linear stability theory for discrete systems. Section 14.7 shows how to find the period $n$ points.


FIGURE 14.4 Transient population response, case 3; oscillates between four values (period 4 behavior).


FIGURE 14.5 Transient population response, case 4; chaotic behavior.


FIGURE 14.6 Transient population response, case 4; chaotic behavior (Solid Line-initial condition of $x_{0}=0.1$. Dashed line--initial condition of $x_{0}=$ 0.101). This illustrates the sensitivity to initial conditions.


FIGURE 14.7 Cobweb diagram for the quadratic map problem. The initiad point is $x_{0}=0.1$.

### 14.4 COBWEB DIAGRAMS

Insight to the behavior of discrete single-variable systems can be obtained by constructing cobweb diagrams. Cobweb diagams are generated by ploting two curves: (i) $g(t)$ versus $x$ and (ii) $x$ versus $x$; the solution (fixed-point) is at the intersection of the two curves. For example, consider the case 1 parameter value of $\alpha=2.95$ and an initial guess, $x_{0}=0.1$. The $x_{n+1}=g(x)=2.95 x_{n}\left(1-x_{n}\right)$ curve is shown as the inverted parabola in Figure 14.7. Since the $x_{0}$ value is 0.1 , the $x_{1}$ value is obtained by first drawing a vertical line to the $g(x)$


FIGURE 14.8 Case $1(\alpha=2.95)$ map, convergence to a single solution $(x=0.661)$; corresponds to the transient response in Figure 14.2.


FIGURE 14.9 Case $2(\alpha=3.20)$ map, oscillates between $x=0.5130$ and 0.7995 after initial transient; initial condition of $x_{0}=0.1$-corresponds to the transient response in Figure 14.3.
curve to find $g\left(x_{0}\right)=0.265$, then drawing a horizontal line to the $x=x$ curve (since $x_{1}=g\left(x_{0}\right)$; therefore, $x_{1}=0.265$ ). A vertical line is drawn to the $g(x)$ curve (to obtain $g\left(x_{1}\right)=0.575$ ), then a horizontal line is drawn to the $x=x$ curve (so, $x_{2}=0.575$ ). These initial steps are shown in Figure 14.7.

Figure 14.8 shows that this process converges to the fixed-point of $x_{\infty}=0.661$, for the case I parameter value of $\alpha=2.95$. Figure 14.9 shows that the iterative process eventually "bounces" betwcen two solutions for the case 2 parameter value of $\alpha=3.2$. This is shown more clearly in Figure 14.10 where an initial guess of $x_{0}=0.5130$ leads to solutions of 0.5130 and 0.7995 (period-2 behavior). Case 3 has period- 4 behavior, as shown in Figures 14.11 and 14.12. Case 4 (Figure 14.13) is an example of chaotic behavior, where the sequence of iterates never repeats.


FIGURE 14.10 Case $2(\alpha=3.20)$ map, oscillates between $x=0.5130$ and 0.7995 ; inital condition of $x_{0}=0.5130$ (period-2 behavior).




FIGURE 14.11 Case $3(x=3.50)$ map, oscillates between $x=0.3828$, $0.8269,0.5009$, and 0.8750 . The initial condition is $x_{0}=0.1$. This corresponds to the transient response in Figure 14.4 (period-4 behavior).

FIGURE 14.12 Case $3(\alpha=3.50)$ map, oscillates between 0.3828 , $0.8269,0.5009$, and 0.8750 , mitial condition of $x_{0}=0.3828$ (period-4 behavior).

FIGURE 14.13 Case $4(\alpha=3.75)$ map, chaotic behavior; corresponds to the transient response in Figure 14.5.

### 14.5 BIFURCATION AND ORBIT DIAGRAMS

When a parameter of a discrete-time model is varied, the number and character of solutions may change; the parameter that is varied is known as a bifurcation parameter. For the quadratic map, $\alpha$ is a bifurcation parameter. We have seen that somewhere between $\alpha=2.95$ and 3.2 , the behavior of the quadratic map changes from asymptotically stable to period- 2 behavior.

A single diagram can be developed that represents the solutions for a large range of $\alpha$ values. We are most interested in the long-term behavior of a system, so for a single $\alpha$ value, we can run a simulation and throw out the initial transient data points (say, the first 250 points). The next points (say, the next 250 ) should then adequately represent the longterm behavior of the system. We can then move on to another value of $\alpha$ and do the same. This is exactly the technique used to generate Figure 14.14, which is an orbit diagram for the quadratic map (see student exercise 7).

### 14.5.1 Observations from the Orbit Diagram (Figure 14.14)

There is a single steady-state solution until $\alpha=3$, where a bifurcation to two solutions occurs. The next bifurcation point is $\alpha=3.44949$, where four solutions emerge. A period- 8 bifurcation oceurs at $\alpha=3.544090$, period- 16 at $\alpha=3.564407$, period- 32 at $\alpha=$ 3.568759 , and period- 64 at $\alpha=3.569692$. Chaos occurs at $\alpha=3.56995$. Notice that there are some interesting "windows" of periodic behavior, after the onset of chaos. For example, at $\alpha=3.83$ we find a window of period- 3 behavior. The period- 3 behavior occurs after approximately 60 time steps, with an initial condition of 0.1 , as shown in Figure 14.15. This behavior is shown more clearly in Figure 14.16 which is simply the data from


FIGURE 14.14 Orbit diagram for the quadratic map. $\alpha$ is the bifurcation parameter.


FIGURE 14.15 Period -3 behavior (after initial transient) for $\alpha=3.83$, initial condition $=0.1$. Periodic values are $x=0.15615,0.50466$, and 0.957417 .


FIGURE 14.16 Period 3 behavior for $\alpha=3.83$. Values are $x=0.15615$, 0.50466 , and 0.957417 .


FICURE 14.17 Period-3 belavior at $\alpha=3.83$. Values are $x=0.15615$,
0.50466 , and 0.957417 .

Figure 14.15 plofted between 75 and 100 time steps. The cobweb diagram of Figure 14.17 also shows the period-3 behavior. Research has shown that period- 3 behavior implies chaotic behavior.

### 14.6 STABILITY OF FIXED-POINT SOLUTIONS

When we performed our case study, we found that case I converged to the predicted fixed point, while the other cases had periodic (or chaotic) solutions that were not attracted to the fixed points. We wish now to use an analytical method to determine when the solutions will converge to the fixed-point solution that is, when is a fixed-point stable? The following stability theorem is identical to the stability theorem used for the numerical analysis in Chapter 3.

Definition Let $x^{*}$ represent the fixed-point solution of $x^{*}=g\left(x^{*}\right)$ or $g\left(x^{*}\right)-x^{*}=0$.

Theorem $x^{*}$ is a stable solution of $x^{*}=g\left(x^{*}\right)$, if $\left|\begin{array}{l}\partial g \\ \partial x\end{array}\right|<1$ when cvaluated at $x^{*}$.

### 14.6.1 Application of the Stability Theorem to the Quadratic Map

Here we will make use of this theorem to determine the stability of a solution to the quadratic map:

$$
\begin{equation*}
g(x)=\alpha x(1-x)=\alpha x-\alpha x^{2} \tag{14.15}
\end{equation*}
$$

So,

$$
\frac{\partial g}{\partial x}=\alpha-2 \alpha x=\alpha(1-2 x)
$$

For simplicity in notation, we will use $g^{\prime}$ to represent $\partial g / \partial x$. Evaluated at $x^{*}$, we have:

$$
\begin{equation*}
g^{\prime}\left(x^{*}\right)=\alpha-2 \alpha x^{*}=\alpha\left(1-2 x^{*}\right) \tag{14.16}
\end{equation*}
$$

Therefore, from (14.16) and the stability theorem, if the following condition is satisfied:

$$
\begin{equation*}
\left|\alpha\left(1-2 x^{*}\right)\right|<1 \tag{14.17}
\end{equation*}
$$

Then: $x^{*}$ is a stable solution.
Remember from (14.14) that there are two solutions to $x^{*}=\alpha x^{*}\left(1-x^{*}\right)$ :

$$
x^{*}=0 \quad \text { or } \quad x^{*}=\frac{\alpha-1}{\alpha}
$$

Momentarily we will generalize the stability results for any value of $\alpha$. First, we will study the four specific cases.

CASE $1 \quad \alpha=2.95$
At one fixed point solution, $x^{*}=0$, we find:

$$
\left|g^{\prime}\left(x^{*}\right)\right|=\left|\alpha\left(1-2 x^{*}\right)\right|=2.95>1
$$

which indicates that the fixed point is unstable.
At the other fixed point solution, $x^{*}=\alpha-1 / \alpha=2.95-1 / 2.95=0.6610$, we find:

$$
\begin{aligned}
\left|g^{\prime}\left(x^{*}\right)\right| & =\left|\alpha\left(1-2 x^{*}\right)\right|=|2.95(1-2(0.6610))|=|-0.9499| \\
& =0.9499<1
\end{aligned}
$$

which assures that the second fixed-point is stable.
For $\alpha=2.95$, we expect the numerical solution to converge to the stable fixed point, $x^{*}=0.6610$, since the other fixed point $\left(x^{*}=0\right)$ is unstable.

The stability results for cases 1 through 4 are compared in Table 14.2. Notice that case 1 is the only one of the four cases where there exists a stable solution. The reader should verify that an initial guess of $x$ arbitrarily close to zero (but not exactly 0), will not converge to zero for any of the four cases.

TABLE 14.2 Stability Results for Cases 1-4

| Case | $\alpha$ | $x^{*}$ | $\left\|g^{\prime}\left(x^{*}\right)\right\|$ | condition | $x^{*}$ | $\left\|g^{\prime}\left(x^{*}\right)\right\|$ | condition |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2.95 | 0 | 2.95 | unstable | 0.6610 | 0.9499 | stable |
| 2 | 3.20 | 0 | 3.20 | unstable | 0.6875 | 1.2000 | unstable |
| 3 | 3.50 | 0 | 3.50 | unstable | 0.7143 | 1.5000 | unstable |
| 4 | 3.75 | 0 | 3.75 | unstable | 0.7333 | 1.7500 | unstable |

### 14.6.2 Generalization of the Stability Results for the Quadratic Map

Notice that we have been quite limited in our study, since we have only considered four cases with $2.95 \leq \alpha \leq 3.75$. Now we will consider the general results for any $\alpha>0$.

Again, recall that there are two fixed-point solutions for a given value of $\alpha$

$$
x^{*}=0 \quad \text { or } \quad x^{*}=\frac{\alpha-1}{\alpha}
$$

At the risk of complicating our notation, let

$$
x_{0}^{*}=0 \quad \text { and } \quad x_{\mathrm{i}}^{*}=\frac{\alpha-1}{\alpha}
$$

Our goal is to determine how the stability of $x_{0}^{*}$ or $x_{1}^{*}$ changes as a function of $\alpha$.

## STABILITY OF $x_{0}^{*}$ AS A FUNCTION OF $\alpha$

Since $x_{0}^{*}=0$ and $g^{\prime}\left(x_{0}^{*}\right)=\alpha-2 \alpha 0=\alpha$
then $\left|g^{\prime}\left(x_{0}^{*}\right)\right|=|\alpha|$
Also, since $\left|g^{\prime}\right|<1$ is required for stability, then $x_{0}^{*}$ is a stable solution only as long as $-1<\alpha<$ J. Otherwise, $x_{0}^{*}$ is unstable (recall that $\alpha<0$ does not make physical sense).

## STABILITY OF $x_{1}^{*}$ AS A FUNCTION OF $\alpha$

Since $x_{1}^{*}=(\alpha-1) / \alpha$ and $g^{\prime}\left(x_{1}^{*}=\alpha-2 \alpha(\alpha-1) / \alpha=\alpha-2(\alpha-1)=-\alpha+2\right.$
then $\left|g^{\prime}\left(x_{1}^{*}\right)\right|=|-\alpha+2|$
which indicates stability for $1<\alpha<3$. Otherwise, $x_{j}^{*}$ is unstable.
These results are shown on the bifurcation diagram of Figure 14.18 for $0<\alpha<4$. Generally, solid lines will be used to represent stable solutions and dotted lines will be used to represent unstable solutions. As discussed above, a change of stability for $x_{0}^{*}$ occurs at $\alpha=1$. Also, changes of stability for $x_{1}^{*}$ occur at $\alpha=1$ and $\alpha=3$. The values of $\alpha$ where the stability characteristics change are known as bifurcation points. The bifurcation that occurs at $\alpha=1$ is commonly known as a "Iranscritical" bifurcation (see Chapter 15) -an exchange of stability between the two solutions has occured.

Notice that Figure 14.18 is a bifurcation diagram based on a linear stability analysis. It differs from an orbit diagram (such as Figure 14.14), because it does not show the periodic behavior obtained from solving the nonlinear algebraic equation for the population growth model. An orbit diagram camot display unstable solutions, however.

### 14.6.3 The Stability Theorem and Qualitative Behavior

The theorem states that if $|\partial g / \partial x|$ at the fixed point, the fixed point is stable. Further, if $\partial g / \partial x$ is negative, then the fixed point solution is oscillatory. If $\partial g / \partial x$ is positive, the be-


FIGURE 14.18 Bifurcation diagram based on linear stability aralysis.
havior is monotonic. We can then develop the following table of results from the stability theorem

| $\frac{\partial g}{\partial x}$ | stability | response |
| :--- | :--- | :--- |
|  | $-1<-1$ | unstable |
| $0<0<\frac{\partial g}{\partial x}<1$ | stable | oscillatory |
|  | $>1$ | stable |

Although the linear stability analysis is useful for determining if a fixed-point is stable, it cannot be used directly to understand possible periodic behavior. This is the topic of the next section.

### 14.7 CASCADE OF PERIOD-DOUBLINGS

We have noted that there appears to be a series of period doublings in route to chaos. The limitation to the method presented in Section 14.6 showed that it could predict that a particular fixed point was unstable, but could not identify the type of periodic behavior that might occur. In this section we will show how to find these period-doubling bifurcation points and the respective branches shown in Figure 14.13.

### 14.7.1 Period-2

When period doubling occurs, the population value at time step $k$ is equal to the value at time step $k-2$. This can be represented by

$$
\begin{equation*}
x_{k}=x_{k-2} \tag{14.18}
\end{equation*}
$$

or

$$
x_{k+2}=x_{k}
$$

using the notation

$$
\begin{equation*}
x_{k+1}=g\left(x_{k}\right) \tag{14.19}
\end{equation*}
$$

then

$$
\begin{align*}
& x_{k+2}=g\left(x_{k+1}\right)  \tag{14.20}\\
& x_{k+2}=g\left(g\left(x_{k}\right)\right)  \tag{14.21}\\
& x_{k+2}=g^{2}\left(x_{k}\right) \tag{14.22}
\end{align*}
$$

Warning: Do not confuse the $g^{2}\left(x_{k}\right)$ notation with that of the square of the operator $\left|g\left(x_{k}\right)\right|^{2}$.

For the quadratic map, we can develop the relationship shown in (14.22):

$$
\begin{equation*}
x_{k+2}=\alpha x_{k+1}\left(1-x_{k+1}\right) \tag{14.23}
\end{equation*}
$$

and substituting $x_{k+1}=\alpha x_{k}\left(1-x_{k}\right)$ into (14.23), we find:

$$
\begin{equation*}
x_{k+2}=\alpha\left[\alpha x_{k}\left(1-x_{k}\right)\right]\left[1-\left(\alpha x_{k}\left(1-x_{k}\right)\right)\right] \tag{14.24}
\end{equation*}
$$

Since (from 14.19):

$$
x_{k 12}=x_{k}
$$

we can write (14.24) as

$$
\begin{equation*}
x_{k}=\alpha\left[\alpha x_{k}\left(1-x_{k}\right)\right]\left[1-\left(\alpha x_{k}\left(1-x_{k}\right)\right)\right] \tag{14.25}
\end{equation*}
$$

Expanding (14.25),

$$
\begin{equation*}
x_{k}=\alpha^{2}\left[-\alpha x_{k}^{4}+2 \alpha x_{k}^{3}-(1+\alpha) x_{k}^{2}+x_{k}\right] \tag{14.26}
\end{equation*}
$$

or

$$
\begin{equation*}
g^{2}\left(x_{k}\right)=\alpha^{2}\left[-\alpha x_{k}^{4}+2 \alpha x_{k}^{3}-(1+\alpha) x_{k}^{2}+x_{k}\right] \tag{14.27}
\end{equation*}
$$

Notice that there are four solutions to the fourth-order polynomial. We can find the solutions graphically by plotting $g^{2}(x)$ versus $x$, as shown in Figure 14.19 (for $\alpha=3.2$ ).

If you closely observe the plot, you will find the following four solutions for period2 behavior:

$$
x^{*}=0,0.5130,0.6875, \text { and } 0.7995
$$

We can see graphically that the solutions $x^{*}=0$ and 0.6875 are unstable, since the slope of $g^{2}\left(x^{*}\right)$ is greater than 1 . (A period-2 solution is stable if $\left|\partial\left(g^{2}(x)\right) / \partial x\right|<1$.) Also, notice that the solution for $x=g(x)$ will always appear as one of the solutions for $x=g^{2}(x)$. If a solution for $x=g(x)$ is unstable, it will also be unstable for $g^{2}(x)$. We can see that $x=0.6875$ is the solution for both $x=g(x)$ and $x=g^{2}(x)$, by observing Figure 14.20.

At this point it is worth showing the results of $x$ versus $g(g(x))$ for case 1, which we know has a single, asymptotically stable solution. Figure 14.21 shows that there is a single stable solution of $x=0.6610$. This makes sense, because as $k \rightarrow \infty$, we know that $x_{k}=0.6610$; this means that $x_{k+2}=x_{k+1}=x_{k}=0.6610$.

We can see from Figure 14.22 that $\alpha=3.0$ is a bifurcation point, since absolute values of the slope of $g(x)$ and $g^{2}(x)=1$ at $x=0.66667$. Figure 14.22 is cleatly a transition point between Figure 14.21 and Figure 14.20.


FIGURE 14.19 Plot of $g(g(x))$ versus $x$ to find the period- 2 values for $\alpha=3.2$.


FIGURE 14.20 Plots of $g(x)$ and $g^{2}(x)$ versus $x$ for $\alpha=3.2$.


FIGURE 14.21 Plots of $g(x)$ and $g(g(x))$ versus $x$ for $\alpha=2.95$. No period-2 behavior.


FIGURE 14.22 Plot of $g(x)$ and $g^{2}(x)$ versus $x$ for $\alpha=3.0$.

### 14.7.2 Period-4

When period-4 behavior occurs, the population value at time step $k$ is equal to the value at time step $k-4$. This can be represented by:

$$
\begin{equation*}
x_{k+4}=x_{k} \tag{14.28}
\end{equation*}
$$

Using the same arguments that we used for period-2 behavior, we can find that since $x_{k+1}=g\left(x_{k}\right)$,

$$
\begin{align*}
x_{k+4} & =g\left(x_{k+3}\right)  \tag{14.29}\\
& =g\left(g\left(x_{k+2}\right)\right) \\
& =g\left(g\left(g\left(x_{k+1}\right)\right)\right) \\
& =g\left(g\left(g\left(g\left(x_{k}\right)\right)\right)\right) \\
x_{k+4} & =g^{4}\left(x_{k}\right) \tag{14.30}
\end{align*}
$$

We can obtain the solutions to (14.30) by plotting $g^{4}(x)$ versus $x$ as shown in Figure 14.23 for $\alpha=3.5$. Again, do not confuse $g^{4}(x)$ with $[g(x)]^{4} . g^{4}(x)$ is an eighth-order polynomial with eight solutions as shown in Figure 14.23.

Figure 14.24 shows that the solutions for $x=g(x)$ and $x=g^{2}(x)$ are also solutions (although unstable) for $x=g^{4}(x)$.


FIGURE 14.23 Plots of $g^{4}(x)$ versus $x$ to find the period-4 values for $\alpha=3.5$.


FIGURE 14.24 Plots of $g(x), g^{2}(x)$ and $g^{4}(x)$ versus $x$ for $\alpha=3.5$.

### 14.7.3 Period-n

By analogy to the period- 2 and period 4 behavior, we can see that for any period $n$, we have the following relationship

$$
\begin{align*}
& x_{k+n}=x_{k}  \tag{14.31}\\
& x_{k+n}=g^{\prime \prime}\left(x_{k}\right) \tag{14.32}
\end{align*}
$$

Note that $\xi^{n}\left(x_{k}\right)$ will be a polynomial that is order $2 n$, and there will be $2 n$ solutions, $n$ of which are stable.

### 14.7.4 Feigenbaum's Number

The quadratic map exhibits a period doubling route to chaos. As the bifurcation parameter $\alpha$ is increased, model goes through a series of period doublings (period-2, period-4, period-8, period-16, etc.). Feigenbaum noticed that the quadratic map had a consistent change in the bifurcation parameter between each period doubling. Indeed, he found that any "one-hump" (see any plot of $g(x)$ for the guadratic map) model will have a cascade of bifurcations which will yield the Feigenbaum number. The Feigenbaum number is calculated by comparing $\alpha$ values at each successive bifurcation point in the following fashion

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \frac{\alpha_{i}-\alpha_{i} 1}{\alpha_{i+1}-\alpha_{i}}=4.669196223 \tag{14.33}
\end{equation*}
$$

where $\alpha_{i}$ represents the parameter value at the $i^{\text {th }}$ period doubling point, where the period is $n=2^{i}$. To obtain a rough estimate of the Feigenbaum number, use the values of $\alpha$ for period- $16\left(2^{4}\right)$, period- $32\left(2^{5}\right)$ and period- $64\left(2^{6}\right)$

$$
\frac{\alpha_{5}-\alpha_{4}}{\alpha_{6}-\alpha_{5}}=\frac{3.568759-3.564407}{3.569692-3.568759}=4.6645
$$

which is close to 4.6692
A summary of the bifurcation points is provided in Table 14.3.
Chaos occurs when the period is $\infty$ (state sequence never repeated) at $\alpha=3.56995$.

TABLE 14.3 Values of a at Bifurcation Points

| $i$ | period | $\alpha$ |
| :---: | :---: | :--- |
| 1 | 2 | 3.0 |
| 2 | 4 | 3.44949 |
| 3 | 8 | 3.544090 |
| 4 | 16 | 3.564407 |
| 5 | 32 | 3.568759 |
| 6 | 64 | 3.569692 |
| $\infty$ | $\infty$ | 3.56995 |

### 14.8 FURTHER COMMENTS ON CHAOTIC BEHAVIOR

We have used the quadratic map (a model of population growth) to introduce you to nonlinear dynamic behavior. This model consisted of a single discrete nonlinear equation. Dynamic behavior similar to period-2 can result from a set of two nonlinear ordinary differential equations. Examples of period behavior in continuous systems include the Lotka-Volterra model used to predict the populations of predator and prey species. The change in a bifurcation parameter that causes a limit-cycle to form in a 2 ODE system is known as a Hopf bifurcation, and will be covered in Chapter 16. Chaos is possible in a set of thee autonomous nonlinear ordinary differential equations. This behavior was discovered by Lorenz in a simple (reduced-order) model of a weather system (really a model of natural convection heat transfer) and will be detailed in Chapter 17. Lorenz coined the phrase "butterfly effect" to describe a system of equations that is sensitive to initial conditions (hence chaotic). He stated conceptually that a butterfly flapping its wings in Troy, New York could cause a monsoon in China several months later (or something similar!).

Some of the earliest results of what is now known as chaos were really discovered by Poincare in the late ninetcenth century, involving the three-body problem. He found that it was easy to determine the planetary motions due to gravity in a system with two bodies, but when three bodies were considered, the system of equations became noninte-grable-leading to the possibility of chaos.

We see turbulence throughout our daily lives, from the water flowing from our faucets, to the effect of wind blowing through our hair as we ride our bicycles, to the boiling water on our stoves. Many researchers have tried to model turbulence by adding stochastic (random) terms to our models of physical behavior. It has only been realized in the past three decades that a good physical (nonlinear) model can simulate the effects of turbulence through chaos.

Numerical methods are used to solve the vast magority of chemical process models. Angelo Lucia (see references) has found that chaos can occur in the solution of some thermodynamic equations of state if the numerical methods are not formulated carefully. It is likely that many people have obtained similarly bad solutions before reformulating them correctly.

## SUMIMARY

A lot of material has been presented in this chapter. You may be wondering how discrete maps and bifurcation theory ties in with applications in chemical engineering. There are at least two important reasons for studying this material:

- We have shown that the quadratic map problem is conceptually identical to numerical methods that can be used to solve a nonlinear algebraic equation. Since the qua dratic map problem exhibits exotic behavior under certain values of the parameter $\alpha$. this tells us that a poorly posed numerical method may have similar problems. Be careful when using numerical methods!
- We noticed that a discrete population growth model is represented by the quadratic map problem. This population growth model is a simple example of a discrete dynamic system, which was modeled by a nonlinear algebraic equation. In the future, we will be studying continuous dynamic systems, that is, systems that are modeled by ordinary differential equations (ODEs). It turns out that nonlinear ODEs can have dynamic properties that are similar to the discrete population model. For example, exothermic chemical reactors can exhibit bifurcation behavior and continuous oscillations in temperature and composition. One main difference is that a system modeled by a set of autonomous ODEs must have at least three equations before chaotic behavior occurs. Chaotic behavior can occur in a discrete model with only one equation.


## REFERENCES AND FURTHER READING

The primary reference for the behavior of the quadratic map is (this has been reprinted in a number of sources) is by May.

May, R.M. (1976). Simple mathematical models with very complicated dynamics, Nature 261: 459-467.

The general fied of chaos theory was introduced to much of the public in the popular book by Gleick:

Gleick, J. (1987). Chaos: Making a New Science. New York: Viking.
Lorenz is given credit for the discovery of "sensitivity to initial conditions":
Lorenz, E.N. (1963). Deterministic nonperiodic flows. Journal of Amospheric: Science, 20: 130-141.

Software for the Macintosh ${ }^{\left.()^{1}\right)}$ that was packaged with the following book was used to generate some of the quadratic map diagrams. This book also does an excellent job of discussing the quadratic map and dynamical systems theory.

Tufillaro, N.B., T. Abbott, \& J. Reilly. (1992). An Experimental Approach to Nonlinear Dynamics and Chaos. Redwood City, CA: Addison-Wesley.

Chaos can appear in numerical solutions to chemical engineering problems, such as phase equilibrium calculations, as shown in the following paper:

Lucia, A., X. Gno, P.J. Richey, \& R. Derebail. (1990). Simple process equations, fixed point methods, and chaos. American Institute of Chemical Engineers Journal (AIChEJ.), 36(5): 641-654.

The book by Strogatz is an excellent introduction to nonlinear dynamics:
Strogatz, S.H. (1994). Nonlinear Dynamics and Chaos. Reading, MA: Addison Wesley.

## STUDENT EXERCISES

1. Why are the results for the simple quadratic map problem important to understand, when chemical and environmental process models are obviously much more complex (based on ODEs)?
2. Use MATLAB to generate transient responses for the quadratic map, for various values of $\alpha$. Explore regions of single steady-state solutions, as well as regions of periodic and chaotic behavior. Use Figure 14.14 to try and find regions of periodic behavior in the midst of chaotic behavior.
3. Derive the scaled logistic equation (14.9) from the following unscaled model for population growth.

$$
n_{k+1}=n_{k}+r\left(1-\frac{n_{k}}{L}\right) n_{k}
$$

where $r$ and $L$ are constants (Hint: Define the scaled variable, $x=n r /(1+r) L$ ). What is the physical significance of $L$ ?
4. Consider the "constant harvesting" model for population growth, where $\gamma$ is a term that accounts for a constant removal rate per unit time period (e.g., hunting deer or removing cells from a petri dish),

$$
x_{k+1}=\alpha x_{k}\left(1-x_{k}\right)-\gamma
$$

How does $\gamma$ effect the equilibrium population values? (Show calculation, and consider stability of the equilibrium.)

Let $\alpha=3.2$. What $\gamma$ values are required for 0 (the trivial solution) to be a stable equibrium solution? What $\gamma$ values are required for a stable nontrivial solution'?
5. Consider the "proportional harvesting" model for population growth, where the removal rate per unit time period is proportional to the amount of population

$$
x_{k+1}=\alpha x_{k}\left(1-x_{k}\right)-\gamma x_{k}
$$

How does $\gamma$ effect the equilibrium population values? (Show calculation, and consider stability of the equilibrium.)

Let $\alpha=3.2$. What $\gamma$ values are required for 0 (the trivial solution) to be a stable equibrium solution? What $\gamma$ values are required for a stable nontrivial solution?
6. Consider the period-2 behavior that occurs at a value of $\alpha=3.2$. Show that the values of $x=0$ and $x=0.6875$ are unstable. (Hint: Let $h(x)=g(g(x))$ and show that $\left|h^{\prime}(x)\right| \geq 1$ at those values.)
7. Using MATLAB construct the orbit diagram (Figure 14.14) for the quadratic map.
8. Find the (real) fixed points of $x_{k+1}=\sqrt{x_{k}}$ and analyze their stability. Also, develop a cobweb diagram for this problem.
9. Consider the nonlinear algebatic equation, $f(x)=-x^{2}-x+1=0$. Using the direct substitution method, formulated as $x=-x^{2}+1=g(x)$, the iteration sequence is

$$
x_{k+1}=g\left(x_{k}\right)=-x_{k}^{2}+1
$$

Try several different initial conditions and show whether these converge, diverge or oscillate between values. Discuss the stability of the two solutions $x^{*}=0.618$ and $x^{*}=1.618$, based on an analysis of $g^{\prime}\left(x^{*}\right)$. Develop a cobweb diagram for this system.
10. Consider the nonlinear algebraic equation, $f(x)=-x^{2}-x+1=0$. Using Newton's method,

$$
x_{k+1}=x_{k}-\frac{f\left(x_{k}\right)}{f^{\prime}\left(x_{k}\right)}
$$

write the iteration sequence in the form of:

$$
x_{k 11}=g\left(x_{k}\right)
$$

Try several different initial conditions and show whether these converge, diverge, or oscillate between values. Discuss the stability of the two solutions $x^{*}=0.618$ and $x^{*}=1.618$, based on an analysis of $g^{\prime}\left(x^{*}\right)$. Develop a cobweb diagram for this system.
11. Consider the scaled Lotka-Volterra (predator $\left(y_{2}\right)$-prey $\left(y_{1}\right)$ ) equations, where

$$
\begin{aligned}
\frac{d y_{1}}{d t} & =\alpha\left(1-y_{2}\right) y_{1} \\
\frac{d y_{2}}{d t} & =-\beta\left(1-y_{1}\right) y_{2}
\end{aligned}
$$

The parameters are $\alpha=\beta=1.0$ and the mitial conditions are $y_{1}(0)=1.5$ and $y_{2}(0)=0.75$. The time unit is days. Integrate these equations mumerically (using ode 45 , for example) to show the periodic behavior.
12. The Henon map is a discrete model that can exhibit chaos:

$$
\begin{aligned}
& x_{1}(k+1)=x_{2}(k)+1-a x_{1}(k)^{2} \\
& x_{2}(k+1)=b x_{1}(k)
\end{aligned}
$$

For a value of $b=0.3$, perform numerical simulations for various values of $a$. Try to find values of $a$ (try $a>0.3675$ ) that yield stable period-2 behavior. Show that chaos occurs at approximately $a=1.06$.
13. Read the paper by Lucia et al. (1990) and use cobweb diagrams to show different types of periodic behavior that can occur when direct substitution is used to find the volume roots of the SRK equation-of-state for the multicomponent mixture $\left(\mathrm{CH}_{4}\right.$, $\mathrm{C}_{2} \mathrm{H}_{4}$, and $\mathrm{C}_{3} \mathrm{H}_{6} \mathrm{O}$ ).

## APPENDIX: MATLAB M-FILES USED IN THIS MODULE

```
    function [time,x] = pmod(alpha,xinit,n)
    population model (quadratic map), pmod.m
    29 August 1993 (c) B.W Bequette
    revised 20 Dec }9
    imput data:
        alpha : growth parameter (between 0 and 4)
        n : number of time steps
        xinit : initial population (between 0 and 1)
    clear x; clear k; clear time;
    x(1) = xinit;
    time(1)= 0;
    Eor k = 2:n+1;
        time(k) = k-1;
        x(k) = alpha*x(k-1)*(1-x(k-1));
    end
% run this file by entering the following in the command
    window
        [time,x] = pmod(alpha,xinit,n);
    with proper values for alpha, xinit and n
    then enter the following
    plot(time,x)
```

    function \([x, g, g 2, g 3, g 4]=g n_{\text {quap }}(a l p h a) ;\)
    $\%$
finds $g(x), g^{\wedge} 2(x), g^{\wedge} 3(x)$ and $g^{\wedge} 4(x)$ functions for
the quadratic map problem
(c) B.W. Bequette
23 july 93
modified 12 Aug 93
revised 20 Dec 96
$\%$
$x=z \operatorname{eros}(201,1) ;$
$g=x ; g 2=x ; g 3=x ; g 4=x$;
$\%$
Eor $i=1: 201 ;$
$x(i)=(i-1) * 0.005$;
$g(i)=a l p h a * x(i) *(1-x(i)) ;$
$g 2(i)=a l p h a * g(i) *(1-g(i)) ;$

$$
\begin{aligned}
& g 3(i)=a l p h a * g 2(i) *(1-g 2(i)) ; \\
& g 4(i)=a 1 p h a * g 3(i) *(1-g 3(i)) ;
\end{aligned}
$$

end
\% can plot, for example
\% plot( $\left.x, x, x, g, x, g 2,{ }^{\prime-} ', x, g 4,{ }^{\prime}-. '\right)$

## BIFURCATION BEHAVIOR OF SINGLE ODE SYSTEMS

The goal of this chapter is to introduce the student to the concept of bifurcation behavior, applied to systems modeled by a single ordinary differential equation. Chapters 16 and 17 will involve systems with more than one state variable.

After studying this chapter, the student should be able to

- Determine the bifurcation point for a single ODE
- Determine the stability of each branch of a bifurcation diagram
- Determine the number of steady-state solutions near a bifurcation point

The major sections in this chapter are:

### 15.1 Molivation

15.2 Illustration of Bifurcation Behavior
15.3 Types of Bifurcations

### 15.1 MOTIVATION

Nonlinear systems can have "exotic" behavior such as multiple steady-states" and transitions from stable conditions to unstable conditions. In Chapter 14 we presented the quadratic map (logistic map or population model), which showed how a discrete-time system could move from a single stable steady-state to periodic behavior as a single parameter was varied. This would be considered a dynamic bifurcation of a discrete-time system, where the behavior changed from asymptotically stable to periodic.

In this chapter we introduce bifurcation behavior of continuous-time systems. A steady-state bifurcation occurs if the number of steady-state solutions changes as a system parameter is changed. If the qualitative (stable vs. unstable) behavior of a system changes as a function of a parameter, we also refer to this as bifurcation behavior. This chapter deals with systems modeled by a single ordinary differential equation.

Bifurcation analysis is particularly important for complex systems such as chemical and biochemical reactors. Although only single variable examples are used in this module, the same types of bifurcation behavior are also observed in chemical and biochemical reactors.

### 15.2 ILLUSTRATION OF BIFURCATION BEHAVIOR

Here a simple polynomial equation will be used to illustrate what is meant by bifurcation behavior. Assume that the following cubic polynomial equation describes the steady-state behavior of a system.

$$
f(x, \mu)=\mu x-x^{3}=0
$$

The solution can be obtained by plotting the function and finding the values of $x$ where $f(x, \mu)=0$. A plot of this function for $\mu=-1,0$ and 1 is shown in Figure 15.1 below. We see that the number of real solutions $(f(x, \mu)=0)$ for $\mu=-1$ is one, while the number of real solutions for $\mu=1$ is three. The curve for $\mu=0$ is a transition between the two cases. We say that $\mu=0$ is a bifurcation point for this system, because the number of real solutions changes from one to three at this point.

We will see in the next section that this behavior is characteristic of a pichfork bifurcation. We will also find that the number of solutions is always three for this problem; sometimes two of the solutions are complex, and other times the solutions are all the same value.


FIGURE 15.1 Polynomial behavior as a function of $\mu$.

### 15.3 TYPES OF BIFURCATIONS

The types of bifurcations that will be presented by way of examples include: (i) pitchfork, (ii) saddle-node, and (iii) transcritical. We will also cover a form of hysteresis behavior and show that it involves two saddle-node bifurcations. Before we cover these specific bifurcations, we will present the general analysis approach.

Consider the general dynamic equation:

$$
\begin{equation*}
\dot{x}=\int(x, \mu) \tag{15.1}
\end{equation*}
$$

where $x$ is the state variable and $\mu$ is the bifurcation parameter. The steady-state solution (also known as an equilibrium point) of (15.1) is:

$$
\begin{equation*}
0=f(x, \mu) \tag{15.2}
\end{equation*}
$$

A bifurcation point is where the both the function and its first derivative are zero:

$$
\begin{equation*}
f(x, \mu)=\frac{\partial f}{\partial x}=0 \tag{15.3}
\end{equation*}
$$

Notice that the first-derivative is also the Jacobian for the single-equation model. Also, the cigenvalue is simply the Jacobian for a single equation system, so the eigenvalue is 0 at a bifurcation point. The number of solutions of ( 15.2 ) can be determined from catastrophe theory. Equation (15.2) has $k$ solutions, if the following criteria are satisfied:

$$
\begin{equation*}
f(x, \mu)=0=\frac{\partial f}{\partial x}=\frac{\partial^{2} f}{\partial x^{2}}=\ldots=\frac{\partial^{k-1} f}{\partial x^{k-1}}=0 \tag{15.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial^{k} f}{\partial x^{k}} \neq 0 \tag{15.5}
\end{equation*}
$$

In Example 15.1 this method is applied to a system that exhibits a pitchfork bifuration.

## EXAMPLE 15.1 Pitchfork Bifurcation

Consider the single variable system shown previously in Section 15.2.

$$
\begin{equation*}
\dot{x}=f(x, \mu)=\mu x-x^{3} \tag{15.6}
\end{equation*}
$$

The equilibrium point is:

$$
f(x, \mu)=0=\mu x_{c} \cdots x_{c}^{3}
$$

the solutions to $/(x, \mu)=0$ are

$$
\begin{aligned}
& x_{e 0}=0 \\
& x_{\mathrm{e} 1}=\sqrt{\mu} \\
& x_{e 2}=-\sqrt{\mu}
\end{aligned}
$$

Notice that if $\mu<0$, then $x_{e}=0$ is the only physically meaningful (real) solution, since $V \mu$ is complex if $\mu<0$.

The Jacobian is

$$
\left.\frac{\partial f}{\partial x}\right|_{x, H}=-3 x_{e}^{2}+\mu
$$

Since the Jacobian is a scalar, then the cigenvalue is equal to the Jacobian:

$$
\lambda=-3 x_{e}^{2}+\mu_{\varphi}
$$

If $\lambda<0$, then the system is stable. If $\lambda>0$, then the system is unstable. Now, we can find the stability of the system, as a function of the bifucation parameter, $\mu$.
I. $\mu<0$. The only real equibibrium solution is $x_{0}=0$, so the vatue of the eigenvalue is:

$$
\lambda=\mu_{e}
$$

which is stable, since $\mu<0$.
II. $\mu>0$. For this case, there are three real solutions; we will analyze each one separately. We use the notation $x_{t 0}, x_{e 1}$, and $x_{e 2}$ to indicate the thee different solutions.
a. $x_{c 0}=0$

$$
\lambda=-3 x_{c}^{2}+\mu_{c}=\mu_{c}=\text { unstable }
$$

b. $x_{e 1}=\sqrt{\mu}$

$$
\begin{aligned}
\lambda & =-3 x_{t}^{2}+\mu_{e} \\
& =\cdots 3 \mu_{e}+\mu_{e}=-2 \mu_{e}=\text { stable }
\end{aligned}
$$

c. $x_{e 2}=-\sqrt{\mu}$

$$
\lambda=-3 x_{e}^{2}+\mu_{e}
$$

$$
=-3 \mu_{e}+\mu_{e}=--2 \mu_{e}=\text { sable }
$$

It is common to plot the equilibrium solutions on a bifurcation diagram, as shown in Figure 15.2 . For $\mu<0$, there is a single real solution, and it is stable. For $\mu>0$ there are three real solutions; two are stable and one is unstable. A solid line is used to represent the stable solutions, white a dashed line indicates the unstable solution. Notice that a change in the number of equilibrium solutions and the type of dynamic behavior occured at $\mu=0$-the bifurcation point. The bifurcation poine satisfies the conditions in (15.3):

$$
f\left(x_{i}, \mu_{c}\right)=\mu_{c} x_{t} \cdots x_{p}^{3}=0
$$

and

$$
\left.\frac{\partial f}{\partial x}\right|_{v_{e}}=-3 x_{e}^{2}+\mu_{e}=0
$$

The state and parameter values that satisfy these conditions simulaneously are:

$$
\mu_{r}=0
$$

and

$$
x_{e}=0
$$

The higher-order derivatives at the bifucation point are:

$$
\left.\frac{d^{2} f}{\partial x^{2}}\right|_{x+4}=6 x=0
$$

and

$$
\left.\frac{\partial^{3} f}{\partial x^{3}}\right|_{x \neq}=6 \neq 0
$$

This analysis indicates that the number of solutions is three in the vicinity of the bifutation point (see (15.4) and (15.5)).


EIGURE 15.2 Pitchfork Bifurcation Diagram-Example 15.1.

It should be noted that there are actually three solutions to the steady-state equation throughout the entire range of $\mu$ values. For $\mu<0$, two of the solutions for $x_{e}$ are complex and one is real. For $\mu=0$, all three solutions for $x_{e}$ are zero. For $\mu>0$, all three solutions for $x_{e}$ are real.


FIGURE 15.3 Transient response for Example 15.1, $\mu=-1$.

### 15.3.1 Dynamic Responses

Figure 15.3 shows the transient response for $\mu=-1$ for two different initial conditions; both initial conditions converge to the equilibrium solution of $x=1$. Figure 15.4 shows the transient response for $\mu=1$ for two different initial conditions; the final steady-state obtained depends on the intial condition. Notice that an initial condition of $x_{0}=0$ would theoretically stay at $x=0$ for all time, however, a small perturbation (say $10^{-9}$ ) would eventually cause the solution to go to one of the two stable steady-states.

Example 15.I illustrates pitchfork bifurcation behavior, where a single real (and stable) solution changes to three real solutions. Two of the solutions are stable, while one is unstable. It is easy to find cases where a (subcritical) pitchfork occurs, that is, where a single unstable solution branches to two unstable and one stable solution. For example, consider the system

$$
\dot{x}=f(x, \mu)=\mu x+x^{3}
$$



FIGURE 15.4 Transient response for Example $15.1, \mu=1$. The final steadystate reached depends on the initial condition.

The reader is encouraged to find the bifurcation behavior of this system shown below (see student exercise 5).


Also, a perturbation of the pitchfork diagram can occur with the following system:

$$
\dot{x}=f(x, \mu, u)=u+\mu x-x^{3}
$$

which can have a diagram of the form shown below (see student exercise 7).


## EXAMPLE 15.2 Saddle-Node Bifurcation (Turning Point)

Consider the single vatiable system:

$$
\begin{equation*}
\dot{x}=f(x, \mu)=\mu-x^{2} \tag{15.7}
\end{equation*}
$$

The equilibrium point is:

$$
f(x, \mu)=0=\mu-x_{e}^{2}
$$

The two solutions are:

$$
\begin{aligned}
& x_{e 1}=\sqrt{\mu} \\
& x_{e 2}=-\sqrt{\mu}
\end{aligned}
$$

The Jacobian (and eigenvalue) is:

$$
\left.\frac{\partial f}{\partial x}\right|_{x_{e}+\cdots,}=-2 x_{e}=\lambda
$$

The bifurcation conditions, (15.4) and (15.5), are satisfied for:

$$
\mu_{e}=x_{e}=0
$$

The second derivative is:

$$
\frac{\partial^{2} f}{\partial x^{2}}=-2 \neq 0
$$

which indicates that there are two solutions in the vicinity of the bifurcation point. Now, we can find the stability of the system, as a function of the bifurcation parameter, $\mu$.
I. $\mu<\boldsymbol{0}$. From $X_{e i}= \pm \sqrt{\mu}$, we see that there is no real solution for $\mu<0$.
II. $\mu>0$. There are now two real solutions; we will analyze the stability of each one.
a. For solution 1:

$$
x_{e 1}=\sqrt{\mu}
$$

the eigenvalue is

$$
\lambda=-2 x_{e}=-2 \sqrt{\mu_{e}}
$$

which is stable.
b. For solution 2:

$$
x_{e 2}=-\sqrt{\mu}
$$

the eigenvalue is

$$
\lambda=-2 x_{e}=-2\left(-\sqrt{\mu_{e}}\right)=2 \sqrt{\mu_{e}}
$$

which is unstable.
The bifurcation diagram (saddle-node) is shown in Figure 15.5.


FIGURE 15.5 Saddle-node bifurcation diagram, Example 15.2.

Notice that there are actually two steady-state solutions for $x_{e}$ throughout the entire range of $\mu$. For $\mu<0$ both solutions for $x_{e}$ are complex; for $\mu=0$ both solutions for $x_{e}$ are 0 ; for $\mu>0$ both solutions for $x_{e}$ ate real.

Dynamic Responses. Transient response curves for $\mu=1$ are shown in Figure 15.6, for two different initial conditions. Initial conditions $x_{0}>-1$ converge to a steady-state of $x=1$, while $x_{0}<-1$ approach $x=-\infty$. It should be noted that a consistent physical (or chemical) -based model will not exhibit this sort of unbounded behavior, since the variables will have some physical meaning and will therefore be bounded.


FIGURE 15.6 Transient response for Example $15.2, \mu=1$. Initial conditions of $x_{0}>-1$ converge to a steady-state of $x=1$, while $x_{0}<-1$ "blows up".

## EXAMPLA 15.3 Transcritical Bifurcation

Consider the single variable system:

$$
\begin{equation*}
\dot{i}=f(x, \mu)=\mu x-x^{2} \tag{15.8}
\end{equation*}
$$

The equilibrium point is:

$$
f\left(x_{t}, \mu\right)=0=\mu x_{e}-x_{e}^{2}
$$

The solutions are

$$
\begin{aligned}
& x_{e 1}=0 \\
& x_{e 2}=\mu
\end{aligned}
$$

The Jacobian is

$$
\left.\frac{\partial f}{\partial x}\right|_{x_{n}, \mu_{x}}=\mu-2 x_{e}
$$

The eigenvalue is also

$$
\lambda=\mu-2 x_{e}
$$

The bifurcation point is $f(x, \mu)=\partial f / \partial x=0$, which occurs at $\mu=x_{e}=0$. The second derivative is:

$$
\frac{\partial^{2} f}{\partial x^{2}}=-2 \neq 0
$$

which indicates that there are two equilibrium solutions. Now, we can find the stability of the system, as a function of the bifurcation parameter, $\mu$.
I. $\mu<0$
a. One solution is:

$$
x_{e 1}=0
$$

with an eigenvalue:

$$
\lambda=\mu-2 x_{e}=\mu_{e}
$$

which is stable (since $\mu_{e}$ is negative).
b. This equilibrium solution is:

$$
x_{e 2}=\mu_{e}
$$

which has the eigenvalue:

$$
\lambda=\mu-2 x_{e}=\mu_{e}-2 \mu_{e}=-2 \mu_{e}
$$

which is unstable (since $\mu_{e}$ is negative).
II. $\mu>0$
a. One solution is

$$
x_{e 1}=0
$$

which has the eigenvalue:

$$
\lambda=\mu-2 x_{e}=\mu_{e}
$$

which is unstable.
b. Another solution is:

$$
x_{e 2}=\mu_{e}
$$

which has the eigenvalue

$$
\lambda=\mu-2 x_{e}=\mu_{e}-2 \mu_{e}=-2 \mu_{e}
$$

which is stable.
These results are shown in the bifurcation diagram of Figure 15.7, which illustrates that the number of real solutions has not changed; however, there is an exchange of stability at the bifurcation point.


FIGURE 15.7 Transeritical bifurcation, Example 15.3.

Dynamic Responses. Transient response curves for the transcritical bifurcation are shown in Figures 15.8 and 15.9. Notice that the transient behavior is a strong function of the initial condition for the state variable. For some initial conditions the state variable eventually settles at a stable steady-state, while for other initial conditions the state variable blows up.


FIGURE 15.8 Transient response for Example 15.3, $\mu=-1$. Notice the importance of initial conditions.


FIGURE 15.9 Transient response for Example 15.3, $\mu=1$. Notice the importance of initial conditions.

The next example is significantly different from the previous examples. Here we allow two parameters to vary and determine their effects on the system behavior.

## EXAMPLE 15.4 Hysteresis Behavior

Consider the system:

$$
\begin{equation*}
\dot{x}=f(x, \mu)=u+\mu x-x^{3} \tag{15.9}
\end{equation*}
$$

which has two parameters ( $u$ and $\mu$ ) that can be varied. We think of $u$ as an adjustable input (manipulated variable) and $\mu$ as a design-related parameter. We will construct steady-state input-output curves by varying $\mu$ and maintaining $\mu$ constant. We will then change $\mu$ and see if the character of the input-output curves ( $x$ versus 10 ) changes. We first work with the case $\mu=-1$.
f. $\mu=-1$. The equilibrium point (steady-state solution) is:

$$
\begin{equation*}
f\left(x_{e}, \mu\right)=0=u-x_{e}-x_{e}^{3} \tag{15.10}
\end{equation*}
$$

The steady-state input-output diagram, obtained by solving (15.10) is shown in Figure 15.10. This curve is generated easily by first generating an $x_{e}$ vector, then solving $u=x_{e}+x_{p}^{3}$.

The stability of each point is found from:

$$
\left.\frac{\partial f}{\partial x}\right|_{v_{s} u_{c}}=-1-3 x_{e}^{2}
$$

which is always negative, indicating that there are no bifurcation points and that all equilibrium points are stable for this system. Contrast this result with that for $\mu=1$, shown next.


FIGURE 15.10 Input-output diagram for Example 15.4 for $\mu=-1$.
II. $\mu=1$. The equilibrium point (steady-state solution) is:

$$
\begin{equation*}
f\left(x_{e}, u\right)=0=u+x_{e}-x_{e}^{3} \tag{15.11}
\end{equation*}
$$

Notice that this is a cubic equation that has three solutions for $x_{e}$ for each value of $u$. For example, consider $u=0$.

At $u=0$ :

$$
x_{e}-x_{p}^{3}=0
$$

so,

$$
x_{e}=1,0, \text { or } 1 .
$$

The stability of each solution can be determined from the Jacobian:

$$
\left.\frac{\partial f}{\partial x}\right|_{x_{c} u_{e}}=1-3 x_{e}^{2}
$$

The eigenvalue is then $\lambda=1-3 x_{e}^{2}$. For the three solutions, we find:

$$
\begin{array}{lll}
x_{e}=-1, & \lambda=1-3=-2 & \text { which is stable. } \\
x_{e}=0, & \lambda=1 & \text { which is unstable. } \\
x_{e}=1, & \lambda=1-3=-2 & \text { which is stable. }
\end{array}
$$

Now we can vary the input, $u$, over a range of values and construct a steady-state input-output curve. These results are shown on the diagram of Figure 15.10 (the easiest way to generate this figure is to create an $x_{e}$ vector, and then solve $u_{e}=-x_{e}+x_{e}^{3}$. See student exercise 2).


FIGURE 15.11 Inptit-output diagram for Example 15.4 with $\mu=1$.

Notice that Figure 15.11 contains two sadde-node (or turning point) bifurcation points (see Example 15.2). The bifurcation (singular) points can be detemined from the solution of:

$$
\begin{equation*}
\left.\frac{\partial f}{\partial x}\right|_{x_{s}, u_{c}}=0=1-3 x_{c}^{2} \tag{15.12}
\end{equation*}
$$

The bifurcation points are then:
or,

$$
\begin{align*}
& x_{c}^{2}=\frac{1}{3} \\
& x_{c}= \pm \frac{1}{\sqrt{3}} \tag{15.13}
\end{align*}
$$

which can be seen to be the $x$ values at the upper and lower tuming points. Substituting (15.13) into ( 15.11 ), we find that the bifurcation points occur at the input values of $u_{c}= \pm 2 / 3 \sqrt{3}$, as shown in Figure 15.10. Notice that for $u<-2 / 3 \sqrt{3}$ or $u>2 / 3 \sqrt{3}$ there is only a single, stable solution, while for $-2 / 3 \sqrt{3}<a<2 / 3 \sqrt{3}$ there are three solutions; two are stable and one is unstable.

We have referred to the behavior of this example as hysteresis behavion now let us show why.

Starting at Low Values of $\mathbf{u}$. Notice that if we begin with a low value of $u$ (say, -3) a single, stable, steady-state value is achieved. If we increase 4 a stight amount (to say, -2.9 ), we will achieve a slightly higher steady-state value for $x$. As we keep increasing $u$, we will continue to achieve a new stable steady-state value for $x$ for each $u$. This continues until $u=2 / 3 \sqrt{3}$, where we find that the stable solution "jumps" to the top curve. Again, as we slowly increase $u$, the stable steady-state solution remains on the top curve.

Starting at High Values of $u$. Notice that if we begin with a high value of $u$ (say, 3) a single stable steady-state value is achieved. If we decrease $u$ a slight amount (to say, 2.9), we will achieve a slightly lower steady-state value for $x$. As we keep decreasing $u$, we will continue to achieve a new stable steady-state value for $x$ for each $u$. This continues until $u=-2 / 3 \sqrt{3}$ where we find that the stable solution "jumps" to the bottom curve. As we slowly decrease $u$ further, the stable steady-state solution remains on the bottom curve.

This is termed hysteresis behavior, because the trajectory (path) taken by the state variable ( $x$ ) depends on how the system is started-up. A jump discontinuity occurs at each "limit" or "turning" point (the saddle-node bifurcation points).

Discussion. Notice that there is a significant difference between the input-output behavior exhibited in Figures 15.10 and 15.11. For $\mu=-1$ (Figure 15.9), there is monotonic relationship between the input ( $u$ ) and the output $(x)$. For $\mu=1$ (Figure 15.10), there is a region of multiplicity behavior, where there are three values of the output $(x)$ for a single value of the input ( $u$ ). There has been a qualitative change in the behavior of this system as $\mu$ varies from -1 to 1 . The value of $\mu$ where this occurs is a hystersis bifurcation point. At this point the following conditions are satisfied (since there are three solutions in the vicinity of the bifurcation point):

$$
f(x, \mu)=0=\frac{\partial f}{\partial x}=\frac{\partial^{2} f}{\partial x^{2}}=0
$$

and

$$
\frac{\partial^{3} f}{\partial x^{3}} \neq 0
$$

The equations are:

$$
\begin{array}{ll}
\dot{x} & =f(x, \mu)=u+\mu x-x^{3}
\end{array}=000 口 \begin{array}{ll}
\left.\frac{\partial f}{\partial x}\right|_{x_{e} x_{e}}=\mu-3 x_{e}^{2} & =0 \\
\left.\frac{\partial^{2} f}{\partial x^{2}}\right|_{x_{e}, u_{e}}=-6 x_{e} & =0 \\
\left.\frac{\partial^{3} f}{\partial x^{3}}\right|_{x_{e}, u_{0}}=-6 & \neq 0
\end{array}
$$

It is easy to show that, for a value of $u=0$, the bifurcation conditions are satisfied at:

$$
x_{e}=\mu_{e}=0
$$

The steady-state input-output curve for this situation is found by solving:

$$
f\left(x_{e}, u\right)=0=u-x_{e}^{3}
$$

which yields the plot in Figure 15.12, which is clearly a transition between Figures 15.10 and 15.11.

A three-dimensional plot of $x$ versus $u$ as a function of $\mu$ is shown in Figure 15.13. The behavior represented by this diagram is commonly known as a cusp catastrophe. At low values


FIGURE 15.12 Input-output diagram for Example 15.4.
of $\mu$ we observe monotonic input-output behavior, with a transition to multiplicity (hysteresis) behavior at $\mu=0$.

The turning points in Figure 15.13 can be projected to the $\mu-u$ plane to find the bifurcation diagram shown in Figure 15.14. A saddle-node (turning point) bifurcation occurs all along the boundary of the regions, except at the "cusp point" ( $\mu=0, u=0$ ), where a codimension- 2 bifurcation occurs. The term "codimension-2" means that two parameters ( $\mu, u$ ) are varied to


FIGURE 15.13 "Cusp catastrophe" diagram for Example 15.4.
achieve this bifucation (Strogatz, 1994). The reader is encouraged to construct this diagram (see student exercise 6).


FIGURE 15.14 Two-parameter $(\mu, u)$ bifurcation diagram for Example 15.4.

## SUMMARY

We have studied the bifurcation behavior of some example single nonlinear ordinary differential equations of the form $\dot{x}=f(x, \mu)$, where $x$ is the state variable and $\mu$ is the bifurcation parameter. The equilibrium (steady-state) points are found by solving $f\left(x_{e}, \mu_{e}\right)=0$. The stability is determined by finding the eigenvalue, $\lambda$, which is simply the Jacobian. $\partial /\left.\partial x\right|_{x_{e}, \mu,}$, for a single equation system. If $\lambda$ is negative, the equilibrium point is stable. If $\lambda$ is positive, the cquilibrium point is unstable.

A bifurcation diagram is drawn by plotting the equilibrium value of the state variable as a function of the bifurcation parameter. If the equilibrium point is stable ( $\lambda=\partial / / \delta I_{r_{c}-\mu_{c}}$ $<0$ ), a solid line is drawn. If the equilibrium point is unstable, a dashed line is drawn. The bifurcation points can be found by solving for $\partial / /\left.\partial x\right|_{x_{e}, x_{c}}=0$ where $f\left(x_{c}, \mu_{c}\right)=0$.

These same techniques can also be applied to systems of several equations, particularly if the equations can be reduced to a single steady-state algebraic equation (in a single state variable). This can be done for many simple chemical and biochemical reactor problems.

## REFERENCES AND FURTHER READING

The following texts provide nice introductions to bifurcation behavior:

Hale, J.K., \& H. Kocak. (1991). Dynamics and Bifurcations. New York: SpringerVerlag.

Jackson, E.A. (1991). Perspectives of Nonlinear Dynamics. Cambridge, UK: Cambridge University Press.

Strogatz, S.H. (1994), Nonlinear Dynamics and Chaos. Reading, MA: AddisonWesley.

## STUDENT EXERCISES

1. For the system in Example 15.4:

$$
\dot{x}=f(x, \mu):=u+\mu x-x^{3}
$$

with $u=0$ and $\mu=1$, perform transient response simulations (using MATLAB) to show that the final steady-state obtained depends on the initial condition.
2. For the system in Example 15.4, we found that there are ranges of $u$ where there are three equilibrium solutions for $x$ (when $\mu=1$ ). When solving for the roots of a cubic polynomial, either a complex analytical solution (see any math handbook) or a root solving routine (such as the MATLAB routine roots) must be used. Show how $x$ can be considered the independent variable and $u$ the dependent variable to obtain an easier analysis of this problem. Then, simply plot $x$ versus $u$.
3. For the system in Example 15.4:

$$
\dot{x}=f(x, \mu)=u+\mu x-x^{3}
$$

with $\mu=1$, show that the saddle-node bifurcation conditions are satisfied at the "turning points."
4. Consider the constant harvesting model of population growth (Hale \& Kocak, 1991):

$$
\dot{x}=f(x, k, c, h)=k x-c x^{2}-h
$$

where all of the parameters are positive. $h$ is the rate of harvesting, while $k$ and $c$ are intrinsic growth rate parameters.

The problem is, for fixed $k$ and $c$, to determine the effect of the barvesting on the population. Since the population density cannot be negative, we are interested in solutions where $x \geq 0$. For a positive initial population density $\left(x_{0}\right)$ the population is exterminated if there is a finite value of $t$ such that $x=0$. Without finding explicit solutions of the differential equation, show the following:
a. If $h$ satisfies $0<h \leq k^{2} / 4 c$, then there is a threshold value of the initial size of the population such that if the initial size is below the threshold value, then the population is exterminated. If the initial size is above the threshold value, then the population approaches an equilibrium (steady-state) point.
b. If $h$ satisfies $h>k^{2} / 4 c$, then the population is exterminated regardless of its initial size.
c. Comment on the physical ramifications of parts a and $b$. Should models be used by State Fish and Game authorities to determine proper hunting and fishing limits?
5. Show that the following system exhibits a pitchfork bifucation, with three real solutions (one stable, two unstable) for $\mu<0$ and a single unstable real solution for $\mu>0$.

$$
\dot{x}=f(x, \mu)=\mu x+x^{3}
$$


6. Consider the system shown in Example 15.4:

$$
\dot{x}=f(x, \mu)=u+\mu x-x^{3}
$$

Develop the cusp bifurcation diagram shown below. Find the values of $u$ and $\mu$ on the boundaries between the one and three fixed-point solution behavior.

7. Consider the system shown in Example 15.4:

$$
\dot{x}=f(x, \mu, u)=u+\mu x-x^{3}
$$

For a value of $u=1$, develop the steady-state bifurcation diagram shown below. Find the values of $x$ and $\mu$ where the saddle-node (turning point) bifurcation oceurs. Notice that this is a perturbation of a pitchfork bifurcation. This type of behavior can occur, for example, in exothermic chemical reactors when the feed flowrate is varied while maintaining a constant jacket temperature (a so-called isola forms).


## APPENDIX

```
cusp diagram
b.w. bequette
15 dec 96
solves the problem
    f(x,u, mut = u + mu*x }x=\mp@subsup{x}{}{\wedge}3=
with }x\mathrm{ varying between -2 and 2
mu varying from -2 to 2 and
whatever u's result
clear x;
cleax u;
clear: mu;
x =- -2:0.05:2;
u = x.^3 + '2*x;
plot3(u,-2*ones(size(u)),x,'w')
hold on
```

```
    mu = -1.875:0.125:2;
    for i = 1:32;
    u = x.^3 - mu(i).*x;
    plot3(u,mu(i)*ones(size(u)),x,'w')
    end
    hold off
>view(15,-30)
```


## BIFURCATION BEHAVIOR OF TWO-STATE SYSTEMS

The goal of this chapter is to introduce the reader to limit cycle behavior and the Hopf bifurcation. After studying this chapter, the reader should be able to

- Find that many of the same types of bifurcations that occur in single-state systems also occur in two-state systems (pitchfork, saddle-node, transcritical)
- Understand the difference between limit cycles (nonlincar behavior) and centers (linear behavior)
- Distinguish between stable and unstable limit cycles
- Determine the conditions for a Hopf bifurcation (formation of a limit cycle)
- Discuss the differences between suberitical and supercritical Hopf bifurcations

The major sections in this chapter are:

### 16.1 Background

16.2 Single-Dimensional Bifurcations in the Phase-Plane
16.3 Limit Cycle Behavior
16.4 The Hopf Bifurcation

### 16.1 BACKGROUND

In Chapter 15 we presented the bifurcation behavior of single-state systems. We found that a number of interesting bifurcation phenomena could occur in these systems, including transcritical, pitchfork, and saddle-node bifurcations. We find in this chapter that these
types of bifurcations can also occur in higher-order systems. This is the subject of Section 16.2. In Section 16.3 we review limit cycle behavior, which was initially presented in Chapter 13 (phase-plane analysis). In Section 16.4 we present a type of bifurcation that can only occur in second- and higher-order systems. In a Hopf bifurcation, we find that a stable node can bifurcate to a stable limit cycle if a parameter is varied; this is an example of a supercritical Hopf bifurcation. This phenomena has been shown to occur in a number of chemical and biochemical reactors. Before turning to the interesting Hopf bifurcation phenomena, we will discuss single dimensional bifurcations in the phase plane.

### 16.2 SINGLE-DIMENSIONAL BIFURCATIONS IN THE PHASE-PLANE

Consider the two-variable system (notice that the two equations are decoupled):

$$
\begin{gather*}
\dot{x}_{1}=f_{1}(x, \mu)=\mu x_{1}-x_{1}^{3}  \tag{16.1}\\
\dot{x}_{2}=f_{2}(x, \mu)=-x_{2} \tag{16.2}
\end{gather*}
$$

The equilibrium (steady-state or fixed-point) solution is:

$$
\mathbf{f}(\mathbf{x}, \mu)=\left[\begin{array}{c}
\mu x_{1 e}-x_{1 e}^{3} \\
-x_{2 e}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

There are three solutions to $f(x, \mu)=0$. The trivial solution is:

$$
\mathbf{x}_{e}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

and the two nontrivial solutions are:

$$
\mathbf{x}_{e}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{c}
\sqrt{\mu} \\
0
\end{array}\right]
$$

and

$$
\mathbf{x}_{c}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{c}
-\sqrt{\mu} \\
0
\end{array}\right]
$$

Notice that only the trivial solution exists for $\mu<0$, since we will assume that equilibrium values must be real (not complex).

We can determine the stability of each equilibrium point from the Jacobian, which is:

$$
\mathbf{A}=\left[\begin{array}{cc}
\mu-3 x_{1 e}^{2} & 0 \\
0 & -1
\end{array}\right]
$$

which has the following eigenvalues:

$$
\begin{aligned}
& \lambda_{1}=\mu-3 x_{1 e}^{2} \\
& \lambda_{2}=-1
\end{aligned}
$$

Since the second eigenvalue is always stable, the stability of each equilibrium point is determined by the first cigenvalue. Here we consider three cases, $\mu<0, \mu=0$, and $\mu>0$.

I $\mu<0$
The only equilibrium solution is the trivial solution ( $x_{1 e}=0$ ), so:

$$
\lambda_{1}=\mu-3 x_{1 e}^{2}=\mu
$$

which is stable, since $\mu<0$.

II $\mu=0$
The equilibrium solution is $x_{1 e}=0$, so:

$$
\lambda_{1}=\mu-3 x_{1 e}^{2}=0
$$

which is stable; the system can be shown to exhibit a slow approach to equilibrium by observing the analytical solution to the differential equations.

III $\mu>0$
The eigenvalue for the trivial solution $\left(x_{1 e}=0\right)$ :

$$
\lambda_{1}=\mu-3 x_{1 e}^{2}=\mu
$$

is unstable since $\mu>0$.
The eigenvalues for the nontrivial solutions $( \pm \sqrt{ } \mu)$ are:

$$
\left(\text { for } x_{1 e}=\sqrt{\mu}\right) \lambda_{1}=\mu-3 x_{e e}^{2}=\mu-3 \mu=-2 \mu
$$

and

$$
\left(\text { for } x_{1 e}=\sqrt{\mu}\right) \lambda_{1}=\mu-3 x_{1 e}^{2}:=\mu-3 \mu=-2 \mu
$$

So the nontrivial solutions are stable for $\mu>0$. This means that a saddle point (trivial solution) is bounded by two stable nodes for this case, since the three solutions are:

$$
\begin{aligned}
& x_{e}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{c}
-\sqrt{\mu} \\
0
\end{array}\right] \text { with } \lambda=\left[\begin{array}{l}
\lambda_{1} \\
\lambda_{2}
\end{array}\right]=\left[\begin{array}{c}
-2 \mu \\
-1
\end{array}\right]=\text { stable node } \\
& x_{e}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{c}
0 \\
0
\end{array}\right] \text { with } \lambda=\left[\begin{array}{c}
\lambda_{1} \\
\lambda_{2}
\end{array}\right]=\left[\begin{array}{c}
\mu \\
-1
\end{array}\right]=\text { saddle point } \\
& x_{e}=\left[\begin{array}{c}
x_{1 e} \\
x_{2 e}
\end{array}\right]=\left[\begin{array}{c}
\sqrt{\mu} \\
0
\end{array}\right] \text { with } \lambda=\left[\begin{array}{l}
\lambda_{1} \\
\lambda_{2}
\end{array}\right]=\left[\begin{array}{c}
-2 \mu \\
-1
\end{array}\right]=\text { stable node }
\end{aligned}
$$

We notice the following phase-plane diagrams (Figure 16.1) as $\mu$ goes from negative to positive.


FIGURE 16.1 Pitchfork bifurcation behavior in the plane. There is a single stable node for $\mu<0$, and two stable ( 0 ) nodes and a saddle point ( $t$, unstable) for $\mu>0$.

### 16.3 LIMIT CYCLE BEHAVIOR

In Chapter 13 we noticed that linear systems that had eigenvalues with zero real portion formed centers in the phase-plane. The phase-plane trajectories of the systems with centers depended on the initial condition values, as shown in Figure 16.2 below. Different initial conditions lead to different closed-cycles.

In this section, and the rest of this chapter, we are interested in limit cycle behavior, as shown in Figure 16.3. The major difference in center (Figure 16.2) and limit cycle


FIGURE 16.2 Example of center behavior.


FIGURE 16.3 Example of limit cycle behavior.
(Figure 16.3) behavior is that limit cycles are isolated closed orbits. By isolated, we mean that a perturbation in initial conditions from the closed cycle eventually returns to the closed cycle (if it is stable). Contrast that with center behavior, where a perturbation in initial condition leads to a different closed cycle.

## EXAMPLE 16.1 A Stable Limit Cycle

Consider the following system of equations, based on polar coordinates:

$$
\begin{align*}
& \dot{r}=r\left(1-r^{2}\right)  \tag{16.3}\\
& \dot{\theta}=-1 \tag{16.4}
\end{align*}
$$

Notice that these equations are decoupled, that is, the value of $r(t)$ is not required to find $\theta(t)$ and vice versa. The second equation indicates that the angle is constantly decreasing. The stability of this system is then determined from an analysis of the first equation.

The steady-state solution of the first equation yields two possible values for $r$. The trivial solution is $r=0$ and the nontrivial solution is $r=1$.

The Jacobian of the first equation is:

$$
\frac{\partial f}{\partial r}=1-3 r^{2}
$$

We see then that the trivial solution $(r=0)$ is unstable, because the eigenvalue is positive $(+1)$. The nontrivial solution is stable, because the eigenvalue is -2 . Any trajectory that starts out close to $r=0$ will move away, while any solution that starts out close to $r=1$ will move towards $r=1$. The time domain behavior for $x_{1}$ is shown in Figure 16.4. Notice that we have converted the states to rectangular coordinates ( $x_{1}=r \cos \theta, x_{2}=r \sin \theta$ ). The phase-plane behavior is shown in Figure 16.5. Initial conditions that are either "inside" or "outside" the limit cycle converge to the limit cycle.


FIGURE 16.4 Stable limit cycle behavior (Example 16.1).


FIGURE 16.5 Stable limit cycle behavior (Example 16.1).

The previous example was for a stable limit cycle. It is also possible for a limit cycle to be unstable, as shown in Example 16.2.

## EXAMPLE 16.2 An Unstable Limit Cycle

Consider the following system of equations, based on cylindrical coordinates

$$
\begin{align*}
& \dot{r}=-r\left(1-r^{2}\right)  \tag{16.5}\\
& \dot{\theta}=-1 \tag{16.6}
\end{align*}
$$

Again, notice that these equations are decompled. The second equation indicates that the angle is constantly decreasing. The stability of this system is then determined from an analysis of the first equation.


FIGURE 16.6 Phase-plane behavior for an unstable limit cycle.

The steady-state solution of the first equation yields two possible values for $r$. The trivial solution is $r=0$ and the nontrivial solution is $r=1$.

The Jacobian of the first equation is:

$$
\frac{\partial f}{\partial r}=-1+3 r^{2}
$$

We see then that the trivial solution is stable, because the eigenvalue is negative ( -1 ). The nontrivial solution is unstable, because the eigenvalue is positive ( +2 ). Any trajectory that starts out less than $r=1$ will converge to the origin, while any solution that starts out greater than $r=1$ will increase at an exponential rate. This leads to the phase-plane behavior shown in Figure 16.6. The time domain behavior is shown in Figure 16.7.


FIGURE 16.7 Time domain behavior for an unstable limit cycle. An initial condition of $r(0)=0.9$ converges to 0 , while an intial condition of $r(0)=1.05$ blows up.

Examples 16.1 and 16.2 have shown the existence of two different types of limit cycles. In the first case (16.1) the limit cycle was stable, meaning that all trajectories were "attracted" to the limit cycle. In the second case (16.2) the limit cycle was stable, and all trajectories were "repelled" from the limit cycle. Although both of these examples yielded limit cycles that were circles in the plane, this will not normally be the case. Usually the limit cycle forms more of an ellipse. Now that we have covered limit cycle behavior, we are ready to determine what types of system parameter changes will cause limit cycle behavior to occur. That is the subject of the next section.

### 16.4 THE HOPF BIFURCATION

In Chapter 15 we studied systems where the number of steady-state solutions changed as a parameter was varied. The point where the number of solutions changed was called the bifurcation point. We also found that an exchange of stability generally occurred at the bifurcation point.

A Hopf bifurcation occurs when a limit cycle forms as a parameter is varied. In the next example we show a supercritical Hopf bifurcation, where the system moves from a stable steady-state at the origin to a stable limit cycle (with an unstable origin) as a parameter is varied.

## EXAMPLE 16.3 Supercritical Hopf Bifurcation

Consider the system:

$$
\begin{align*}
& \dot{x}_{1}=x_{2}+x_{1}\left(\mu-x_{1}^{2}-x_{2}^{2}\right)  \tag{16.7}\\
& \dot{x}_{2}=-x_{1}+x_{2}\left(\mu-x_{1}^{2}-x_{2}^{2}\right) \tag{16.8}
\end{align*}
$$

This can be written (see student exercise 4) in polar coordinates as:

$$
\begin{align*}
& \dot{r}=r\left(\mu-r^{2}\right)  \tag{16.9}\\
& \dot{\theta}=-1 \tag{16.10}
\end{align*}
$$

Since these equations are decoupled, the stability is determined from the stability of:

$$
\dot{r}=f(r)=r\left(\mu-r^{2}\right)
$$

the Jacobian is:

$$
\frac{\partial f}{\partial r}=\mu-3 r^{2}
$$

The equilibrium (steady-state) point is $f(r)=0$, which yields,

$$
r\left(\mu-r^{2}\right)=0
$$

which has three solutions:

$$
\begin{aligned}
& r=0 \text { (trivial solution) } \\
& r=\sqrt{\mu} \\
& r=-\sqrt{\mu} \text { (not physically realizable) }
\end{aligned}
$$

For $\mu<0$, only the trivial solution ( $r=0$ ) exists. For $\mu<0$,

$$
\frac{\partial f}{\partial r}=\bar{x}
$$

which is stable, since $\mu<0$.
For $\mu=0$, all of the steady-state sohtions are $r=0$, and the Jacobian is:

$$
\frac{\partial f}{\partial r}=-3 r^{2}
$$

which is stable, but has slow convergence to $r \approx 0$.
For $\mu>0$, the trivial solution $(r=0)$ is unstable, because:

$$
\begin{aligned}
& \partial f \\
& \partial r
\end{aligned}=\mu
$$

The nontrivial solution $(r=\sqrt{\mu})$ is stable because:

$$
\frac{\partial f}{\partial r}=\mu-3 r^{2}=-2 \mu
$$

and we find the following phase-plane plots shown in Figure 16.8.
The bifurcation-diagram for this system is shown in Figure 16.9.


FIGURE 16.8 Phase-plane plots. As $\mu$ goes from -1 to I, the behavior changes from stable node to a stable Imit cycle.


FIGURE 16.9 Bifurcation diagram. Indicates that the origin $(r=0)$ is stable when $\mu<0$. When $\mu>0$ the origin becomes unstable, but a stable limit cycle (with radius $r=\sqrt{\mu}$ ) emerges.

Here we analyze this system in rectangular $\left(x_{1}-x_{2}\right)$ coordinates. The only steady-state (fixedpoint or equilibrium) solution to (16.7) and (16.8) is:

$$
x_{e}=\left[\begin{array}{l}
x_{1 e} \\
x_{2 k}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

Linearizing (16.7) and (16.8):

$$
\begin{array}{ll}
\frac{\partial f_{1}}{\partial x_{1}}=\mu-3 x_{1}^{2}-x_{2}^{2} & \frac{\partial f_{1}}{\partial x_{2}}=1-2 x_{1} x_{2} \\
\frac{\partial f_{2}}{\partial x_{1}}=-1-2 x_{1} x_{2} & \frac{\partial f_{2}}{\partial x_{2}}=\mu-x_{1}^{2}-3 x_{2}^{2}
\end{array}
$$

We find the Jacobian matrix:

$$
\mathbf{A}=\left[\begin{array}{cc}
\mu-3 x_{1 e}^{2}-x_{2 e}^{2} & 1-2 x_{1 e} x_{2 k} \\
-1-2 x_{1 e} x_{2 e} & \mu-x_{1}^{2},-3 x_{2 e}^{2}
\end{array}\right]
$$

which is, for the equilibrium solution of the origin:

$$
\mathbf{A}=\left[\begin{array}{cc}
\mu & 1 \\
-1 & \mu
\end{array}\right]
$$

The characteristic polynomial, from $\operatorname{det}(\lambda \mathbf{I}-\mathbf{A})=0$, is:

$$
\lambda^{2}-2 \mu \lambda+\mu^{2}+1=0
$$

which has the eigenvalues (roots):

$$
\lambda=\mu \pm \frac{\sqrt{4 \mu^{2}-4\left(\mu^{2}+1\right)}}{2}=\mu \pm 1 j
$$

We see that when $\mu<0$, the complex eigenvalues are stable (negative real portion); when $\mu=0$, the eigenvalues lie on the imaginary axis; and when $\mu>0$, the complex eigenvalues are unstable (positive real portion). The transition of eigenvalues from the left-half plane to the right-half plane is shown in Figure 16.10.


FIGURE 16.10 Location of eigenvalues in complex plane as a function of $\mu$. A Hopf bifurcation occurs as the eigenvalues pass from the lefthand side to the righthand side of the complex plane.

Example 16.3 was for a supercritical Hopf bifurcation, where a stable limit cycle was formed. We leave it as an exercise for the reader (student exercise 6) to show the formation of a subcritical Hopf bifurcation, where an unstable limit cycle is formed.

We have found that the Hopf bifurcation occurs when the real portion of the complex eigenvalues became zero. In Example 16.3 the eigenvalues crossed the imaginary axis with zero slope, that is, parallel to the real axis. In the general case, the eigenvalues will cross the imaginary axis with non-zero slope.

We should also make it clearer how an analysis of the characteristic polynomial of the Jacobian (A) matrix can be used to identify when a Hopf bifurcation can occur. For a two-state system, the characteristic polynomial has the form:

$$
a_{2}(\mu) \lambda^{2}+a_{1}(\mu) \lambda+a_{0}(\mu)=0
$$

where the polynomial parameters, $a_{i}$, are shown to be a function of the bifurcation parameter, $\mu$ (It should also be noted that it is common for $a_{2}=1$ ). Assume that the $a_{i}(\mu)$ parameters do not become 0 for the same value of $\mu$. It is easy to show that a Hopf bifurcation occurs when $a_{1}(\mu)=0$ (see student exercise 7).

### 16.4.1 Higher Order Systems ( $n>2$ )

Thus far we have discuss Hopf bifurcation behavior of two-state systems. Hopf bifurcations can occur in any order system ( $n \geq 2$ ); the key is that two complex eigenvalues cross the imaginary axis, while all other eigenvalues remain negative (stable). This is shown in Figure 16.11 for the three state case.


FIGURE 16.11 Location of eigenvalues in complex plane as a function of $\mu$. A Hopf bifurcation occurs as the eigenvalues pass from the lefthand side to the righthatid side of the complex plane.

## SUMMARY

In this chapter we have shown that the same bifurcations that occured in single-state systems (saddle-node, transcritical, and pitchfork) also oceur in systems with two or more states. We have also introduced the Hopf bifurcation, which occurs when complex eigenvalues pass from the left-half plane to the right-half plane, as the bifurcation parameter is varied. A Hopl bifurcation can also occur in systems with more than two states. For a supercritical Hopf bifurcation, two complex conjugate eigenvalues cross from the left-half to the righthalf plane, while all of the other eigenvalues remain stable (in the left-half plane).

## FURTHER READING

The following sources provide general introductions to bifurcation theory:
Hale, J., \& H. Kocak (1991). Dynamics and Bifurcations. New Yok: SpringerVerlag.
Strogať, S.H. (1994). Nonlinear Dynamics and Chaos. Reading, MA: AddisonWestey.

The following textbook shows a complete example of the occurance of Hopf bifurcations in a 2 -state exothermic chemical reactor model:

Varma, A., \& M. Morbidelli. (1997). Mahematical Methods in Chemical Engineering. New York: Oxford University Press.

## STUDENT EXERCISES

1. Show that the two-variable system

$$
\begin{aligned}
& \dot{x}_{1}=f_{1}(x, \mu)=\mu-x_{1}^{2} \\
& \dot{x}_{2}=f_{2}(x, \mu)=-x_{2}
\end{aligned}
$$

exhibits saddle-node behavior in the phase plane.
2. Show that the two-variable system

$$
\begin{aligned}
& \dot{x}_{1}=f_{1}(x, \mu)=\mu x_{1}-x_{1}^{2} \\
& \dot{x}_{2}=f_{2}(x, \mu)=-x_{2}
\end{aligned}
$$

exhibits transcritical behavior in the phase plane.
3. Show that the two-variable system:

$$
\begin{aligned}
& \dot{x}_{1}=f_{1}(x, \mu)=u+x_{1}-x_{1}^{3} \\
& \dot{x}_{2}=f_{2}(x, \mu)=-x_{2}
\end{aligned}
$$

exhibits hysteresis behavior in the $x_{1}$ state variable. This means that, as $u$ is varied, $x_{1 e}$ follows an S-shaped curve, which exhibits the ignition/extinction behavior shown in Chapter 15.
4. Show that:

$$
\begin{aligned}
& \dot{x}_{1}=x_{2}+x_{1}\left(\mu-x_{1}^{2}-x_{2}^{2}\right) \\
& \dot{x}_{2}=-x_{1}+x_{2}\left(\mu-x_{1}^{2}-x_{2}^{2}\right)
\end{aligned}
$$

can be written:

$$
\begin{aligned}
& \dot{r} \approx r\left(\mu-r^{2}\right) \\
& \dot{\theta}=-1
\end{aligned}
$$

if $x_{1}=r \cos \theta$ and $x_{2}=r \sin \theta$.
5. Consider a generalization of Example 16.3, which was a supercritical Hopf bifurcation (a stable limit cycle);

$$
\begin{aligned}
& \dot{r}=r\left(\mu-r^{2}\right) \\
& \dot{\theta}=\omega+b r^{2}
\end{aligned}
$$

Discuss how $\omega$ affects the direction of rotation. Also, discuss how $b$ relates the frequency and amplitude of the oscillations.
6. Consider the following system, which undergoes a subcritical Hopf bifurcation:

$$
\begin{aligned}
& \dot{r}=\mu r+r^{3}-r^{5} \\
& \dot{\theta}=\omega+b r^{2}
\end{aligned}
$$

Show that, for $\mu<0$ an unstable limit cycle lies in between a stable limit cycle and a stable attractor at the origin. What happens when $\mu=0$ and $\mu>0$ ?
7. Show that the condition for a Hopf bifurcation for the following characteristic equation

$$
a_{2}(\mu) \lambda^{2}+a_{1}(\mu) \lambda+a_{0}(\mu)=0
$$

is $a^{1}(\mu)=0$. This is easy to do if you realize that a Hopf bifurcation occurs when the roots have zero real portion and write the polynomial in factored form.

Relate this condition to the Jacobian matrix, $A(\mu)$, realizing that:

$$
\lambda^{2}-\operatorname{tr}(A(\mu)) \lambda+\operatorname{det}(A(\mu))=0
$$

8. Consider a Hopf bifurcation of a three-state system. Realizing that one pole is negative (and real) and that the other two poles are on the imaginary axis, relate the Hopf bifurcation to the coefficients of the characteristic polynomial of the Jacobian matrix are:

$$
a_{3}(\mu) \lambda^{3}+a_{2}(\mu) \lambda^{2}+a_{1}(\mu)^{1}+a_{0}(\mu)=0
$$

You can assume, without loss of generality, that $a^{3}(\mu)=1$. How do the conditions on the polynomial coefficients relate to the conditions on the Jacobian matrix (trace, determinant, etc.)?

# INTRODUCTION TO CHAOS: THE LORENZ EQUATIONS 

The objective of this chapter is to present the Lorenz equations as an example of a system that has chaotic behavior with certain parameter values. After studying this chapter, the reader should be able to:

- Understand what is meant by chaos (extreme sensitivity to initial conditions)
- Understand conceptually the physical system that the Lorenz equations attempt to model.
- Understand how the system behavior changes as the parameter $r$ is varied.

The major sections in this chapter are:

### 17.1 Introduction

17.2 Background
17.3 The Lorenz Equations
17.4 Stability Analysis of the Lorenz, Equations
17.5 Numerical Study of the Lorenz Equations
17.6 Chaos in Chemical Systems
17.7 Other Issues in Chaos

### 17.1 INTRODUCTION

In Chapter 14 we presented the quadratic map (logistic equations) and found that the transient behavior of the population varied depending on the growth parameter. Recall that when the qualitative behavior of a system changes as a function of a certain parameter, we refer to the parameter as a bifurcation parameter. As the growth parameter was varied, the population model went through a series of period-doubling behavior, finally becoming chaotic at a certain value of the growth parameter. At that time we noted that chaos is possible with one discrete nonlinear equation, but that chaos could only occur in continuous (ordinary differential equation) models with three or more equations (assuming the model is autonomous). In this chapter we study a continuous model that has probably received the most attention in the study of chaos-w-the Lorenz equations. Before we write the equations, it is appropriate to give a brief historical perspective on the Lorenz model. For a more complete history, see the book Chaos by James Gleick (1987).

### 17.2 BACKGROUND

In 1961, Edward Lorenz, a professor of Meteorology at MIT, was simulating a reducedorder model of the atmosphere, which consisted of twelve equations. Included were functional relationships between temperature, pressure, and wind speed (and direction) among others. He performed numerical simulations and found recognizable patterns to the behavior of the variables, but the patterns would never quite repeat. One day he decided to examine a particular set of conditions (parameter values and initial conditions) for a longer period of time than he had previously simulated. Instead of starting the entire simulation over, he typed in a set of initial conditions based on results from midway through the previous run, started the simulation and walked down the hall for a cup of coffee. When he returned, he was shocked to find that his simulation results tracked the previous run for a period of time, but slowly began to diverge, so that after a long period of time there appeared to be no correlation between the runs. His first instinct was to check for a computer error; when he found none, he realized that he had discovered a very important aspect of certain types of nonlinear systems-that of extreme sensitivity to initial conditions. When he had entered the new initial conditions, he bad done so only to a few decimal places, whereas several more decimal places were carried internally in the calculations. This small difference in the initial conditions built up over a period of time, to the point where the two runs did not look similar. This discovery led to the realization that long-term prediction of certain systems (such as the weather) will never be possible, no matter how many equations are used and how many variables are measured.

In order to learn more about the behavior of these types of systems, he reduced his model of the atmosphere to the fewest equations that could describe the bare essentialsthis required three equations. Here we discuss the "physics" of the three equations, while Section 17.3 presents the equations and discusses the equilibrium solutions and stabilty of the equations.


FIGURE 17.1 Convection rolls due to a temperature gradient in a fluid where density decreases as a function of temperature ( $T_{1}>T_{2}$ ).

Consider a fluid maintained between two parallel plates, as shown in Figure 17.1. When the top plate temperature $\left(\mathrm{T}_{2}\right)$ is equal to the bottom plate temperature $\left(\mathrm{T}_{1}\right)$, there is no flow and the system is in equilibrium. Now, slowly increase the bottom temperature. At low temperature differences, there is still no flow because the viscous forces are greater than the buoyancy forces (the tendency for the less dense fluid at the bottom to move toward the top and the more dense fluid to move toward the bottom). Finally, at some critical temperature difference, the buoyancy forces overcome the viscous forces and the fluid begins to move and form convection rolls. As the temperature difference is increased, the fluid movement becomes more and more vigorous. Although the following point may be less clear to the reader, for some systems there is a value of temperature difference that will cause the smooth convection rolls to break up and become turbulent or chaotic.

One can think of the speed of these convection rolls as wind speed in a miniature "weather model" and the direction of the convection rolls as wind direction.

In the next section, we analyze the Lorenz equations, which attempt to model the flow pattern of Figure 17.1.

### 17.3 THE LORENZ EQUATIONS

The Lorenz equations are:

$$
\begin{align*}
& \dot{x}_{1}=\sigma\left(x_{2}-x_{1}\right)  \tag{17.1}\\
& \dot{x}_{2}=r x_{1}-x_{2}-x_{1} x_{3}  \tag{17.2}\\
& \dot{x}_{3}=b x_{3}+x_{1} x_{2} \tag{17.3}
\end{align*}
$$

Notice that the only nonlinear terms are the bilinear terms $x_{1} x_{3}$ in (17.2) and $x_{1} x_{2}$ in (17.3).

The state variables have the following physical significance:
$x_{1}=$ proportional to the intensity (speed) of the convective rolls
$x_{2}=$ proportional to the temperature difference between the ascending and descending currents
$x_{3}=$ proportional to distortion of the vertical temperature profile from linearity

Three parameters, $\sigma, r$, and $b$ have the following physical significance:
$\sigma=$ Prandtl number (ratio of kinematic viscosity to thermal conductivity)
$r=$ ratio of the Rayleigh number, $R a$, to the critical Rayleigh number, $R a_{c}$
$b=$ a geometic factor related to the aspect ratio (height/width) of the convection roll

The Rayleigh number is: $\quad R a=\frac{g \alpha H^{3} \Delta T}{\nu k}$
where:
$\alpha=$ coefficient of expansion
$H=$ distance between plates
$g=$ gravitational accelcration
$\Delta T=$ temperature difference between the plates $\left(T_{1}-T_{2}\right)$
$y=$ kinematic viscosity
$k=$ thermal conductivity
For a fixed geometry and fluid, $R a$ is a dimensionless measure of the temperature difference between the plates. For $0 \leq r<1\left(R a<R a_{c}\right)$ the temperature difference is not large enough for the buoyancy forces to overcome the viscous forces and cause motion. For $r>1\left(R a>R a_{c}\right)$ the temperature difference is large enough to cause motion.

### 17.3.1 Steady-State (Equilibrium) Solutions

The Lorenz equations have three steady-state (equilibrium) solutions under certain conditions. First, we present the trivial solution, then the nontrivial solutions. In Section 17.4 we will determine the stability of each equilibrium solution.

## TRIVIAL SOLUTION

By inspection we find that the trivial solution to (17.1)-(17.3) is $x_{1 s}=x_{2 s}=x_{3 s}=0$
The trivial condition corresponds to no convective flow of the fluid.

## NONTRIVIAL SOLUTIONS

From (17.1) we find that:

$$
\begin{equation*}
x_{1 s}=x_{2 \mathrm{~s}} \tag{17.5}
\end{equation*}
$$

Substituting (17.5) into (17.3), we find that $x_{3 s}=\frac{1}{b} x_{1 s,}^{2}$, or:

$$
\begin{equation*}
x_{1 s}= \pm \sqrt{b x_{3 s}} \tag{17.6}
\end{equation*}
$$

TABLE 17.1 Sumnary of the Equilibrium Solutions

| State Variable | Trivial Solution | Nontrivial I <br> $(r>0$ required $)$ | Nontrivial 2 <br> $(r>0$ required $)$ |
| :---: | :---: | :---: | :---: |
| $x_{1 s}$ | 0 | $\sqrt{b(r-1)}$ | $-\sqrt{b(r-1)}$ |
| $x_{2 s}$ | 0 | $\sqrt{b(r-1)}$ | $-\sqrt{b(r-1)}$ |
| $x_{3 s}$ | 0 | $r-1$ | $r \cdots 1$ |

Substituting (17.6) into (17.2) at steady-state, we find:

$$
\begin{equation*}
x_{3 s}=r-1 \tag{17.7}
\end{equation*}
$$

and substituting (17.7) into (17.6) and using the results of (17.5):

$$
\begin{equation*}
x_{1 s}=x_{2 s}= \pm \sqrt{b(r-1)} \tag{17.8}
\end{equation*}
$$

For real solutions to (17.8), condition $r \geq 1$ must be satisfied. This means that for $r<1$, there is only one real solution (the trivial solution), while for $r>1$ there are three real solutions. This is an example of a pitchfork bifurcation. For $R a<R a_{c}$, there is no convective flow. The equilibrium behavior is summarized in Table 17.1.

### 17.4 STABILITY ANALYSIS OF THE LORENZ EQUATIONS

Linearizing (17.1)-(17.3) around the steady-state, we find the following Jacobian matrix:

$$
A=\left[\begin{array}{ccc}
-\sigma & \sigma & 0  \tag{17.9}\\
r-x_{3 s} & -1 & -x_{1 s} \\
x_{2 s} & x_{1 s} & -b
\end{array}\right]
$$

which we will analyze to determine the stability of the equilibrium solution.

### 17.4.1 Stability of the Trivial Solution

For the trivial solution, $x_{1 s}=x_{2 s}=x_{3 s}=0$, the Jacobian matrix is:

$$
A=\left[\begin{array}{ccc}
-\sigma & \sigma & 0  \tag{17.10}\\
r & -1 & 0 \\
0 & 0 & -b
\end{array}\right]
$$

and the stability is determined by finding the roots of $\operatorname{det}(\lambda I-A)=0$.

$$
\begin{align*}
\lambda I-A & =\left[\begin{array}{ccc}
\lambda+\sigma & -\sigma & 0 \\
-r & \lambda+1 & 0 \\
0 & 0 & \lambda+b
\end{array}\right] \\
\operatorname{det}(\lambda I-A) & \left.=(\lambda+b) \left\lvert\, \begin{array}{cc}
\lambda+\sigma & -\sigma \\
-r & \lambda+1
\end{array}\right.\right]  \tag{17.11}\\
& =(\lambda+b)[(\lambda+\sigma)(\lambda+1)-\sigma r] \\
\operatorname{det}(\lambda-A) & =(\lambda+b)\left[\lambda^{2}+(\sigma+1) \lambda+\sigma(1-r)\right]
\end{align*}
$$

and we see from (17.11) that the eigenvalues are

$$
\begin{align*}
& \lambda_{1}=-b  \tag{17.12}\\
& \lambda_{2}=\frac{-(\sigma+1)-\sqrt{(\sigma+1)^{2}-4 \sigma(1-r)}}{2}  \tag{17,13}\\
& \lambda_{3}=-(\sigma+1)+\sqrt{(\sigma+1)^{2}-4 \sigma(1-r)} \tag{17.14}
\end{align*}
$$

Clearly, the first eigenvalue is always stable, since $b>0$. It is also easy to show that the second and third eigenvalues can never be complex. The second cigenvalue is always negative and the third eigenvalue is only negative for $r<1$. We then see the following eigenvalue structure for the trivial (no flow) solution
$r<0$ : all eigenvalues are negative, trivial solution is stable
$r>0$ : saddle point (one unstable eigenvalue), trivial solution is unstable

For $r>1$, then $R a>R a_{e}$, which means that flow will occur. Notice that $R a$ is proportional to $\Delta T$. This means that once $\Delta T$ is increased beyond a certain critical $\Delta T_{c}$, convective flow will begin.

### 17.4.2 Stability of the Nontrivial Solutions

Here we find the roots of $\operatorname{det}(\lambda I-A)=0$ for the nontrivial solutions. Starting with:

$$
\lambda I-A=\left[\begin{array}{ccc}
\lambda+\sigma & -\sigma & 0  \tag{17.15}\\
x_{3 s}-r & \lambda+1 & x_{1 s} \\
-x_{2 s} & -x_{1 s} & \lambda+b
\end{array}\right]
$$

and solving $\operatorname{det}(\lambda I-A)=0$, you should find:

$$
\begin{equation*}
\lambda^{3}+b_{2} \lambda^{2}+b_{1} \lambda+b_{0} \tag{17.16}
\end{equation*}
$$

where:

$$
\begin{align*}
& b_{2}=-\operatorname{tr}(A)=\sigma+b+1  \tag{17.17}\\
& b_{1}=(r+\sigma) b  \tag{17.18}\\
& b_{0}=-\operatorname{det}(\Lambda)=2 \sigma b(r-1) \tag{17.19}
\end{align*}
$$

Recall that real nontrivial solutions only exist for $r>1$. The coefficients $b_{2}, b_{1}$, and $b_{0}$ are then all positive, satisfying the necessary condition for stability. The Routh array must be used to check the sufficient condition for stability. As derived in the appendix, the critical $r$ for stability is:

$$
\begin{equation*}
r_{i l}=\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \tag{17.20}
\end{equation*}
$$

If $r>r_{H}$, then the stability condition is not satisfied. This is an interesting result, because it means that none of the equibrium solutions (trivial, nontrivial 1, nontrivial 2) is stable for $r>r_{H}$. The subscript $H$ is used in (17.20), because a Hopf bifurcation forms at that value (see student exercise 1). If a supercritical Hopf bifurcation occurred, a stable limit cycle would form, yiedling periodic behavior for the nontrivial solutions. It turns out that a subcritical Hopf birfurcation is formed, that is, the limit cycle is unstable (see Strogatz for a nice discussion). Since there are no stable equilibrium points for $r>r_{f f}$, and no stable limit cycles, the solution "wanders" in phase space, never repeating the same trajectory. This behavior is known as chaos and the solution is said to be a strange attractor.

### 17.4.3 Summary of Stability Results

We have seen that for $r<1$ there is only one real solution, the trivial solution, and it is stable. When $r=1$ there is a pitchfork bifurcation, yielding three real solutions for $r>1$. The trivial solution is unstable for $r>1$, while the nontrivial solutions are stable for $1<r<r_{H}$. This behavior is shown clearly by the bifurcation diagram shown in Figure 17.2. The formation of the unstable limit cycle at $r=r_{u}$ is discussed by Strogatz.


FIGURE 17.2 Bifurcation diagram for the Lorenz equations. Based on parameters in Section 17.5.

### 17.5 NUMERICAL STUDY OF THE LORENZ EQUATIONS

Lorenz used the following values to illustrate the chaotic nature of the equations

$$
\begin{aligned}
\sigma & =10 \\
b & =\frac{8}{3} \\
r & =28
\end{aligned}
$$

from (17.22) we calculate that $r_{H}=470 / 19=24.74$, indicating that all of the equilibrium points are unstable, since the value of $r=28$ is greater than $r_{H}$.

Before we continue with the set of parameters that Lorenz used to illustrate chaotic behavior, we will first perform simulations for two other cases. In the first, we show a set of conditions where the trivial solution is stable. In the second, we show a set of conditions where the nontrivial solutions are stable.

### 17.5.1 Conditions for a Stable Trivial (No Flow) Solution

We have found that the trivial steady-state is stable for $0 \leq r<1$. Here we use the $\sigma$ and $b$ parameters used by Lorenz, but set $r=0.5$ for a stable trivial steady-state.

$$
\sigma=10 \quad b=\frac{8}{3} \quad r=0.5
$$

As in future simulations, we assume an initial condition of $x_{0}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}$. A time domain plot for all three-state variables is shown in Figure 17.3, and a phase-plane plot ( $x_{3}$ vs. $x_{1}$ ) is shown in Figure 17.4. The convergence to equilibrium occurs rapidly.


FIGURE 17.3 Lorenz equations under conditions for a stable trivial solution.


FIGURE 17.4 Phase-plane under stable conditions for the trivial solution ( $x_{1}-x_{3}$ plane).

### 17.5.2 Stable Nontrivial Solutions

We have found that the nontrivial steady-states are stable for $1<r<r_{H}$. Here we use the $\sigma$ and $b$ parameters used by Lorenz, but set $r=10$ (recall that $r_{H}=24.74$ for these values of $\sigma$ and $b$ ) to show that the nontrivial steady-states are stable.

$$
\sigma=10 \quad b=\frac{8}{3} \quad r=21
$$

For the trivial steady-state, $\mathbf{x}=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{T}$, the eigenvalues are:

$$
\begin{aligned}
& \lambda_{1}=-2.67 \\
& \lambda_{2}=-20.67 \\
& \lambda_{3}=9.67
\end{aligned}
$$

as expected, the trivial steady-state is unstable (a saddle point).
The nontrivial steady-states, $\mathbf{x}=\left[\begin{array}{lll}\sqrt{160 / 3} & \sqrt{160 / 3} & 20\end{array}\right]^{T}$ and $\mathbf{x}=[-\sqrt{160 / 3}$ $-\sqrt{160 / 3} \quad 20]^{T}$, have cigenvalues of:

$$
\begin{aligned}
& \lambda_{1}=-13.4266 \\
& \lambda_{2}=-0.1200+8.9123 j \\
& \lambda_{3}=-0.1200-8.9123 j
\end{aligned}
$$

verifying that the nontrivial steady-states are stable.
Time domain plots for $\mathbf{x}_{1}$ are shown in Figure 17.5, for $r=21$ and two different initial conditions. Notice that convergence to a particular equilibrium point depends on the initial condition, that is, plot a converges to one equilibrium point, while plot $b$ converges to a different equilibrium point. Also notice that plot bexhibits what is known as transient


FIGURE 17.5 Lorenz equations under conditions for stable nontrivial solutions.
chaos. The initial trajectory appears chaotic, but eventually the trajectory converges to an equilibrium point. In other systems the system can exhibit transient chaos and settle into periodic behavior.

The phase plane diagram of Figure 17.6 also clearly shows the effect of two different initial conditions. In curve $a$ the trajectory almost immediately goes to the equilibrium point on the right (positive value of $x_{1}$ ). In curve $b$ the trajectory first winds around the left equilibrium point, switches to the right equilibrium point, and (after going back and forth a few times) eventually winds around the left equibrium point, slowly converging.


FIGURE 17.6 Phase plane under stable conditions for the nontrivial solutions.

### 17.5.3 Chaotic Conditions

The parameters used for this case are

$$
\sigma=10 \quad b=\frac{8}{3} \quad r=28
$$

Recall that all of the equilibrium points are unstable, since the value of $r=28$ is greater than $r_{H}$.

For the trivial steady-state, $\mathbf{x}=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{T}$, the eigenvalues are:

$$
\begin{aligned}
& \lambda_{1}=-2.67 \\
& \lambda_{2}=-22.83 \\
& \lambda_{3}=11.83
\end{aligned}
$$

as expected, the trivial steady-state is unstable (a saddle point).
For the nontrivial stcady-states, $\mathbf{x}=\left[\begin{array}{lll}\sqrt{72} & \sqrt{72} & 27\end{array}\right]^{T}$ and $\mathbf{x}=[-\sqrt{72}$ $-\sqrt{72} \quad 27]^{T}$, the eigenvalues are:

$$
\begin{aligned}
& \lambda_{1}=-13.8546 \\
& \lambda_{2}=0.0940-10.1945 j \\
& \lambda_{3}=0.0940+10.1945 j
\end{aligned}
$$

indicating that the nontrivial steady-states are unstable. Notice that all of the steady-state operating points are unstable. This means that the curves in the three-dimensional "phasecube" plots will not asymptotically approach any single equilibrium point. The curves may exhibit periodic-type behavior, where the three-dimensional equivalent of a limit cycle is reached. The curves could even have "quasi-periodic" behavior, where the oscillations appear to have two frequency components. It tums out for this set of parameters that the curves never repeat. The curves have a strange attractor because they stay in a certain region of three-space, but never intersect or repeat. This is known as chaotic behavior.

Figure 17.7 shows the Lorenz behavior for the $x_{1}$ variable under unstable (chaotic) conditions. The initial condition is $x_{0}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}$. Plots of the other states ( $x_{2}$ and $x_{3}$ ) are similiar.

Figure 17.7 was a time domain plot for $x_{1}$ under chaotic conditions. More interesting results are also shown in the following phase-plane diagram (Figure 17.8). Notice that the trajectory will spend some time "winding around" one equilibrium point, before jumping to the other side and winding around the other equilibrium point for a while. This process goes on forever, with the trajectory never crossing itself (in 3-space).


FIGURE 17.7 Transient response of $x_{1}$ under chaotic conditions.


FIGURE 17.8 Phase-plane of Lorenz equations under chaotic conditions.

The development of the curve in Figure 17.8 is shown more clearly in the phaseplane plots in Figure 17.9, which show varies "pieces" of time.

A three-dimensional plot (phase cube) of this trajectory is shown in Figure 17.10.
The reader is encouraged to perform simulations of the Lorenz equations, to be understand concepts such as sensitivity to initial conditions. MATLAB has a demo titled lorenz.m (simply enter lorenz in the command window) that traces a three-dimensional plot of solutions to the Lorenz equations. Each new run uses a new set of random initial conditions. If you write an m-file to simulate the Lorenz equations using ode45, remember to use a name that is different than lorenz.m, to avoid conflicts with the MATLAB demo.

### 17.6 CHAOS IN CHEMICAL SYSTEMS

The Lorenz equations provide a nice example of chaos, because the equations are reasonably simple to analyze. An even simpler set of equations was developed by Rossler to demonstrate chaotic behavior (see student exercise 3). Chaos has also been shown to appear in models of chemical process systems, particularly exothermic chemical reactors. Reactors that are forced periodicatly (jacket temperature is a sine wave, for example) have been shown to exhibit chaos. Also, a series of reactors with heat integration can exhibit chaotic behavior. It appears that chaotic reactors may have had low anplitude "osciltations" (say in temperature) that may have been interpreted as measurement and process noise in the past. A comprehensive review of nonlinear dynamic behavior in chemical reactors is provided in the article by Razon and Schmitz (1987).

$t=0$ to 5
Lemenz Edeations. $t=10$ tot $=15$

$t=10$ to 15

$\mathrm{t}=20 \mathrm{to} 25$

$\mathrm{t}=30$ to 35


$$
t=5 \text { to } 10
$$


$\mathrm{t}=15 \mathrm{t}) 20$

$t=25$ to 30


$$
t=35 \text { to } 40
$$

FIGURE 17.9 Phase-plane of Lorenz equations under chaotic conditions.


FIGURE 17.9 Continued

### 17.7 OTHER ISSUES IN CHAOS

Chaos is a complex field with many books and conferences devoted to this simple topic. Clearly, it is impossible to give this topic adequate coverage in a single chapter. Our goal is to provide an introduction to, and motivation for, the topic. The reader is encouraged to consult the many books and articles available on the topic.

$$
r=28, \quad x_{0}=[0 ; 1 ; 0]
$$



FIGURE 17.10 Three-dimensional phase space plot of Lorenz equations under chaotic conditions.

Issues that may be of particular interest include:

- How does one detect chaos experimentally? One method is to use experimental data to calculate Lyapunov exponents. See Strogatz for example.
- Chaos can be used to encode secret messages. See Cuomo and Oppenheim (1993), who used ideas presented by Pecora and Carroll (1990).


## SUMMARY

We have presented an introduction to chaotic behavior by studying the Lorenz convective flow equations. A number of chemical processes have been shown to exhibit similar behavior. It is necessary to have three nonlinear autonomous differential equations before chaos can occur. Although not shown here, chaos can occur in a system of two nonlincar nonautonomous equations (that is, if some type of periodic input forcing is used). Also, we saw in Chapter 14 that chaos can occur in a single discrete nonlinear equation (the quadratic map, or logistic equation).

## REFERENCES AND FURTHER READING

A nice book on the history of chaos, written for the general public, is by Gleick.
Gleick, J, (1987). Chaos: Making a New Science. New York: Viking.
The first paper to develop the notion of sensitivity to initial conditions is by Lorenz.
Lorenz, E.N. (1963). Deterministic nonperiodic flow. Journal of Atmospheric Sciences, 20: 130-141.

A number of introductory-level textbooks provide nice introductions to chaos. These include:

Strogatz, S.H. (1994). Nonlinear Dynamics and Chaos. Reading, MA: AddisonWesley.

A review of nonlinear dynamic behavior (including Chaos) of chemical reactors is provided by:

Razon, L.F., \& R.A. Schmitz. (1987). Multiplicities and instabilities in chemically reacting systems--A Review. Chem. Eng. Sci., 42(5): 1005-1047.

A large number of papers on chaos have been collect in the following book:
Hao, Bai-Lin. (Ed.). (1990). Chaos I/. Singapore: World Scientific Press.

Papers that develop a way of encoding secret messages using chaos are:

Pecora, L.M., \& T.L. Carroll. (1990). Synchronization in chaotic systems. Physical Review Letters, 64: 821.
Cuomo, K.M., \& A.V. Oppenheim. (1993). Circuit implementation of synchronized chaos, with applications to communications. Physical Review Letters. 71: 65.

## STUDENT EXERCISES

1. Consider the following parameter values for the Lorenz equations:

$$
\begin{aligned}
\sigma & =10 \\
r & =r_{c}=470 / 19=24.74 \\
b & =\frac{8}{3}
\end{aligned}
$$

For the nontrivial solution, show that a supercritical Hopf bifurcation occurs at this value of $r$. That is, for $r<r_{c}$, all eigenvalues are stable, for $r=r_{c}$, two eigenvalues are on the imaginary axis, and for $r>r_{c}$, two eigenvalues have crossed into the right half plane.
2. Show the sensitivity to initial conditions of the Lorenz equations. Run two simulations with the parameter values shown in the numerical study

$$
\begin{aligned}
\sigma & =10 \\
r & =28 \\
b & =8 / 3
\end{aligned}
$$

For the first simulation use the initial condition $x_{0}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}$. For the second simulation use the initial condition $x_{0}=\left[\begin{array}{lll}0 & 1.01 & 0\end{array}\right]^{T}$. When do the simulations begin to diverge?

Run some more simulations with smaller perturbations in the initial conditions. Also, make perturbations in the initial conditions for the other state variables. What do you find?
3. Consider the Rossler equations (see Strogatz, for example):

$$
\begin{aligned}
& \dot{x}_{1}=-x_{2}-x_{3} \\
& \dot{x}_{2}=x_{1}+a x_{2} \\
& \dot{x}_{3}=b+x_{3}\left(x_{1}-c\right)
\end{aligned}
$$

which have a single nonlinear term. Let the parameters $a$ and $b$ be constant with a value of 0.2 . Use simulations to show that this system has period- 1 (limit cycle), pe-riod-2, and period-4 behavior for $c=2.5,3.5$, and 4 , respectively. Show that chaotic behavior occurs for $c=5$.
4. The Henon map is a discrete model that can exhibit chaos:

$$
\begin{aligned}
& x_{1}(k+1)=x_{2}(k)+1-a x_{1}(k)^{2} \\
& x_{2}(k+1)=b x_{1}(k)
\end{aligned}
$$

For a value of $b=0.3$, perform numerical simulations for various values of $a$. Try to find values of $a$ (try $a>0.3675$ ) that yield stable period-2 behavior. Show that chaos occurs at approximately $a=1.06$.

## APPENDIX

Stability analysis of the nontrivial steady-state using the Routh array:

| Row |  |  |
| :---: | :--- | :--- |
| 1 | 1 | $b_{1}$ |
| 2 | $b_{2}$ | $b_{0}$ |
| 3 | $b_{2} b_{1}-b_{0}$ | 0 |
| 4 | $b_{2}$ |  |
| Where | $b_{0}=2 \sigma b(r-1)$ | $b_{1}=(r+\sigma) b$ |

Since the nontrivial steady-state only exists for $r \geq 1$, then $b_{0}$ is always positive. It also follows that $b_{1}$ and $b_{2}$ will always be positive. The only entry from the Routh array that we must check is the first column in row 3. This entry will be positive if:

$$
\begin{equation*}
b_{2} b_{1}-b_{0}>0 \quad \text { or } \quad b_{2} b_{1}>b_{0} \tag{A-1}
\end{equation*}
$$

Making the substitutions for parameter values in the coefficients:

$$
\begin{equation*}
(\sigma+b+1)(r+\sigma) b>2 \sigma b(r-1) \tag{A-2}
\end{equation*}
$$

After some algebra, this can be written:

$$
\begin{equation*}
(-\sigma+b+1) r>-\sigma(\sigma+b+3) \tag{A-3}
\end{equation*}
$$

or,

$$
\begin{equation*}
(\sigma-b-1) r<\sigma(\sigma+b+3) \tag{A-4}
\end{equation*}
$$

and, assuming that $\sigma>b+1$, the condition on $r$ for stability is:

$$
\begin{equation*}
r<\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \tag{A-5}
\end{equation*}
$$

Notice from (A-3) that if $\sigma<b+1$, then any $r$ satisfies the requirement for stability. We will often define the critical value, $r_{c}$ :

$$
\begin{equation*}
r_{c}=\frac{\sigma(\sigma+b+3)}{(\sigma-b-1)} \tag{A-6}
\end{equation*}
$$

If $r>r_{c}$, then the system is unstable.

