TÉCNICAS DE DESCOMPOSICIÓN EN PROGRAMACIÓN MATEMÁTICA

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Montevideo, lunes 5 de Mayo de 2025

HIPÓTESIS DEL CURSO.

- *Los Métodos de Descomposición proveen una técnica numérica de optimización que ha resultado apropiada ante problemas complejos.*

OBJETIVO DEL CURSO.

- *Introducir y formar en el empleo de técnicas algorítmicas de descomposición para la resolución de problemas de programación matemática de alta complejidad y gran escala.*

BIBLIOGRAFIA.

- Bazaraa et al. (1998).
- Bertsimas y Tsitsiklis (1997).
- Fourer et al. (2003).

SISTEMA DE EVALUACIÓN.

- Actividades Complementarias (20%).
- Trabajo Final (80%).

CLASES.

- Online: lunes 5 y lunes 12 de mayo.
- Presenciales: lunes 19 al viernes 23 de mayo, de las 16:00 a 19:00hrs en el salón 726.

CONTENIDOS

1. Introducción a los Métodos de Descomposición.
2. Formulación y resolución de modelos en AMPL.
3. Método de Benders.
4. Generación de Columnas.
5. Método de Dantzig & Wolfe.
6. Conclusiones, Extensiones y palabras finales.

1.- Introducción a los Métodos de Descomposición

Existe una amplia variedad de problemas en diversas disciplinas que dan origen a modelos de optimización estructurados de gran tamaño, donde, por ejemplo, se combinan restricciones simples con otras más complejas.

Un Método de Descomposición es una técnica numérica de optimización que resulta muy apropiada ante problemas de esta naturaleza.

En efecto, hay problemas con restricciones lineales (Ax=b), cuya matriz resulta ser una matriz con muchísimas más columnas que ecuaciones o poco densa, con un claro patrón de sus elementos no-nulos.

Por ejemplo, A podría ser una matriz bloque angular:

$$\begin{bmatrix} A_1 & A_2 & A_3 & \cdots & A_q \\ D_1 & & & & \\ & D_2 & & & \\ & & D_3 & & \\ & & & \ddots & \\ & & & & D_q \end{bmatrix}$$

tener una estructura dual bloque angular:

$$\begin{bmatrix} D_1 & & & & F_1 \\ & D_2 & & & F_2 \\ & & D_3 & & F_3 \\ & & & \ddots & \\ & & & D_q & F_q \end{bmatrix}$$

o bien ser una matriz con forma de escalera:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{ccccccc|c} \boxed{A_1} & & & & & & & b^1 \\ \boxed{A'_2} & \boxed{A_2} & & & & & & b^2 \\ \boxed{A'_3} & & \boxed{A_3} & & & & & b^3 \\ & \boxed{A'_4} & & \boxed{A_4} & & & & b^4 \\ & \boxed{A'_5} & & & \boxed{A_5} & & & b^5 \\ & & \boxed{A'_6} & & & \boxed{A_6} & & b^6 \\ & & \boxed{A'_7} & & & & \boxed{A_7} & b^7 \end{array}$$

Un modelo lineal de Optimización Estocástica con recurso de dos-etapas (en el caso finito) corresponde por ejemplo a:

$$
\begin{array}{lllllll}
\text{Min} & cx + p_1q^1y^1 + p_2q^2y^2 + \cdots + p_Sq^Sy^S & & & & & \\
\text{s.a.} & & & & & & \\
 & Ax & & & & & = b \\
 & T^1x + Wy^1 & & & & & = h^1 \\
 & T^2x & + Wy^2 & & & & = h^2 \\
 & \ldots & & & & & \\
 & T^Sx & & & & + Wy^S & = h^S \\
 & x \geq 0,\ y^1 \geq 0,\ y^2 \geq 0,\ \ldots,\ y^S \geq 0. & & & & &
\end{array}
$$

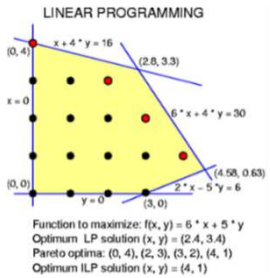
Existen problemas de optimización de gran tamaño altamente estructurados que contribuyen en la toma de decisiones.

Los trabajos seminales en la literatura de métodos de descomposición que explotan este tipo de estructuras surgieron varias décadas atrás:

Dantzig & Wolfe Decomposition (1960).

Column Generation (Gilmore and Gomory, 1961)

Benders Decomposition (1962).



Sus extensiones y aplicaciones hasta hoy son muy variadas y permanentes.

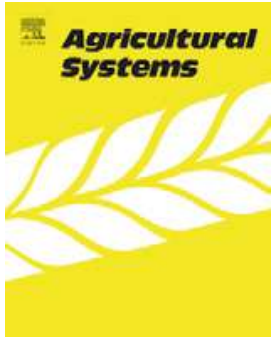
En Albornoz et al. (2004) se abordó un problema de expansión de capacidad de un sistema de potencia eléctrica térmico, que ante incertidumbre en la disponibilidad de las unidades de generación existentes se resolvió con el Método de Descomposición de Benders.

Pérez Canto (2008) formula y resuelve un problema de operación y mantenimiento de unidades de generación en un sistema de potencia eléctrica empleando igualmente el *Método de Benders*.

Azi et al. (2010) formulan y resuelven un problema de ruteo vehicular con ventanas de tiempo y múltiples viajes por vehículo empleando *Generación de Columnas*.

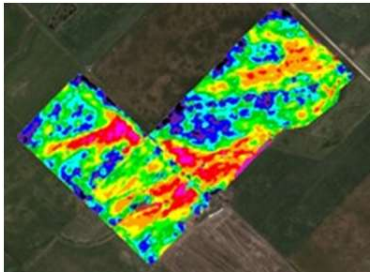
$$
\begin{aligned}
\text{(MP) Min} \quad & \sum_{w \in \Omega} d_w - \alpha g_w) x_w \\
\text{s.t.} \quad & \sum_{w \in \Omega} a_{iw} x_w \leqslant 1, \quad i \in V, \\
& \sum_{w \in \Omega} x_w \leqslant |K|, \\
& x_w \in \{0, 1\}, \quad w \in \Omega.
\end{aligned}
$$

Ahumada et al. (2012) aplicaron el Método de Benders para la búsqueda de un plan óptimo de cultivo y distribución de productos frescos ante rendimientos y demandas bajo incertidumbre.



Costa et al. (2014) aplicaron el método de G*eneración de Columnas* para hallar rotaciones óptimas de cultivo de modo de satisfacer demandas por diferentes productos agrícolas.

En Albornoz y Ñanco (2016) se abordó un problema de partición en zonas homogéneas de un terreno agrícola, basado en información geo-referenciada de propiedades del suelo, mediante un *Método de Generación de Columnas* ante la gran cantidad de variables del modelo.



En Bagger et al. (2018) se resuelve la programación de horarios de asignaturas mediante el Método de Descomposición de Benders.

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Benders' decomposition for curriculum-based course timetabling

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Maximum flow
Minimum cut

ABSTRACT

In this paper we applied Benders' decomposition to the Curriculum-Based Course Timetabling (CBCT) problem. The objective of the CBCT problem is to assign a set of lectures to time slots and rooms. Our approach was based on segmenting the problem into time scheduling and room allocation problems. The Benders' algorithm was then employed to generate cuts that connected the time schedule and room allocation. We generated only feasibility cuts, meaning that most of the solutions we obtained from a mixed integer programming solver were infeasible, therefore, we also provided a heuristic in order to regain feasibility.

We compared our algorithm with other approaches from the literature for a total of 32 data instances. We obtained a lower bound on 23 of the instances, which were at least as good as the lower bounds obtained by the state-of-the-art, and on eight of these, our lower bounds were higher. On two of the instances, our lower bound was an improvement of the currently best-known. Lastly, we compared our decomposition to the model without the decomposition on an additional six instances, which are much larger than the other 32. To our knowledge, this was the first time that lower bounds were calculated for these six instances.

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Invited Review

The Benders decomposition algorithm: A literature review

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Combinatorial optimization
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Acceleration techniques
Literature review

ABSTRACT

The Benders decomposition algorithm has been successfully applied to a wide range of difficult optimization problems. This paper presents a state-of-the-art survey of this algorithm, emphasizing its use in combinatorial optimization. We discuss the classical algorithm, the impact of the problem formulation on its convergence, and the relationship to other decomposition methods. We introduce a taxonomy of algorithmic enhancements and acceleration strategies based on the main components of the algorithm. The taxonomy provides the framework to synthesize the literature, and to identify shortcomings, trends and potential research directions. We also discuss the use of the Benders Decomposition to develop efficient (meta-)heuristics, describe the limitations of the classical algorithm, and present extensions enabling its application to a broader range of problems.

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En Zhang et al (2020) se resuelve el problema de asignación de cirugías programadas a pabellones bajo incertidumbre en los tiempos de duración de las cirugías mediante el Método de *Generación de Columnas*.

Master surgery schedule.

No. of OR	Monday	Tuesday	Wednesday	Thursday	Friday
1	ENT	ENT	ENT		
2			ENT	ENT	ENT
3	OBGYN		OBGYN		OBGYN
4	ORTHO	ORTHO		ORTHO	ORTHO
5		ORTHO		NEURO	
6	GEN	GEN	GEN	GEN	
7		GEN	GEN	GEN	GEN
8	OPHTH	OPHTH		OPHTH	OPHTH
9	VASCULAR		CARDIAC		VASCULAR
10	UROLOGY		ORTHO		

En Albornoz y Zamora (2024) se abordó un problema de definición de zonas de manejo agrícola, basada en información de propiedades del suelo, para la asignación de rotaciones de cultivos orgánicos con restricciones de adyacencia. Para su resolución se empleó *Generación de Columnas* (y filas dependientes).

Existen numerosos problemas de gran tamaño altamente estructurados que contribuyen a la toma de buenas decisiones y cuya resolución plantea desafíos algorítmicos por su complejidad.

La presencia de tal estructura puede y (debe) ser empleada en métodos eficientes para la resolución de tales tipos de problema (ante instancias de gran tamaño).

Para abordar computacionalmente un problema complejo mediante el uso de un Método de Descomposición la idea básica consiste en:

descomponer o simplificar la resolución del problema original resolviendo separadamente un Problema Maestro (Reducido) y un Subproblema, este último usualmente con un subconjunto de las restricciones originales que poseen una estructura especial o de más fácil resolución.

Para llevar a cabo esta idea en un contexto algorítmico, la estrategia consiste en enviar información tales como *precios sombra* y soluciones factibles entre el *Problema Maestro* (*Reducido*) *y el Subproblema* hasta alcanzar la solución óptima o una aproximación a la misma en un número finito de iteraciones.

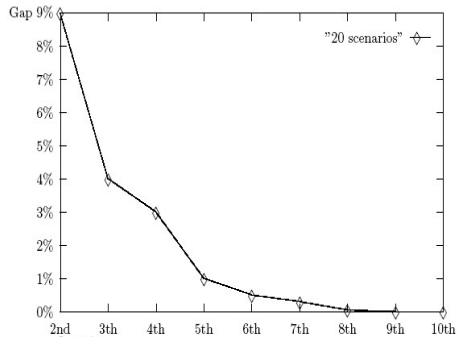


Fig 1. Algorithm behavior.

En Albornoz et al. (2004) se considera el Método de Descomposición de Benders donde:

Problema Maestro: plan de inversiones sobre todo el horizonte de planificación de largo plazo.

Subproblemas: operación óptima del sistema en cada mes y escenario de disponibilidad de unidades.

En Pérez Canto (2008) se considera el Método de Descomposición de Benders donde:

Problema Maestro: plan de mantenimiento sobre todo el horizonte de planificación.

Subproblema: operación óptima del sistema en cada periodo de acuerdo a la disponibilidad de unidades.

En Azi et al. (2010) se considera un:

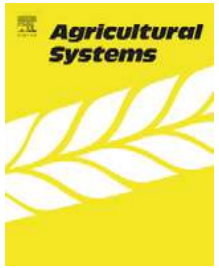
Problema Maestro: Modelo con rutas posibles para los distintos vehículos empleados.

Subproblema: determina una nueva ruta factible.

Ahumada et al. (2012) aplicaron el Método de Benders, donde:

Problema Maestro. Encuentra un plan óptimo de cultivos de productos frescos.

Subproblemas. maximiza los ingresos esperados ante la realización de posibles escenarios de los parámetros inciertos (rendimientos y demandas futuras).



Costa et al. (2014) aplicaron Generación de Columnas, donde:

Problema Maestro. Encuentra un plan óptimo de rotación de cultivos que permita satisfacer demandas sobre un conjunto de tierras de cultivo.

Subproblema. Provee una nueva potencial rotación de cultivo para un terreno.

En Albornoz y Ñanco (2016) se considera el *Método de Generación de Columnas* donde:

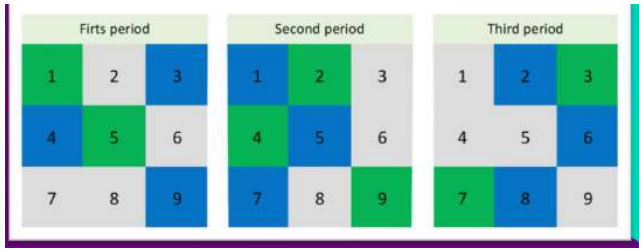
Problema Maestro: Partición óptima del terreno para un conjunto dado (generado) de potenciales zonas de manejo agrícola.

Subproblema: determina una nueva potencial zona rectangular para el terreno.

Albornoz y Zamora (2024) aplicaron Generación de *Columnas* a un problema de agricultura orgánica, donde:

Problema Maestro. Encuentra una partición óptima de un terreno en zonas de manejo, asigna cultivos de vegetales e impone restricciones de adyacencia (entre familias botánicas).

Subproblema. Provee una nueva zona de manejo (rectangular).



Algunas extensiones del método de Benders:

L-Shaped decomposition method of van Slyke & Wets (1969).

Generalized Benders Decomposition of Geoffrion (1972).

Stochastic Decomposition, Higle and Sen (1991).

L-Shaped Method for IPSP, Caroe and Tind (1998).

Benders and BFC for 0-1 mixed SP, Escudero et al. (2007).

Stochastic Scenario Decomposition, Higle et al. (2010)

Cluster Benders Decomposition, Aramburu et al. (2012).

Benders method for bilevel programs, Bagloee et al. (2016), Fontaine and Minner (2017) and Che et al. (2019).

Son numerosos los métodos que extienden GC así como las aplicaciones del mismo, entre otros:

VRPTW, Desrosiers et al. (1984).

Routing with Time Windows by Column Generation*

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Gerad, *Ecole des Hautes Etudes Commerciales, Montreal, Quebec, Canada H3T 1V6*

Column Generation in IP, Vanderveck and Wolsey (1996).

Branch-and-price, Barnhart et al. (1998).

Consider a set of trips where each trip is specified a priori by a place of origin, a destination, a duration, a cost, and a time interval within which the trip must begin. The trips may include visits to one or more specific points. Our problem is to determine the number of vehicles required, together with their routes and schedules, so that each trip begins within its given time interval, while the fixed costs related to the number of vehicles, and the travel costs between trips, are minimized. The problem is a generalization of the *m*-traveling salesman problem. We use column generation on a set partitioning problem solved by simplex and branch-and-bound; columns are generated by a shortest path algorithm with time windows on the nodes. Numerical results for several school bus transportation problems with up to 151 trips are discussed.

Two-Stage Column Generation, Salani and Vacca (2008).

Primal-dual column generation, Gondzio *et al*. (2013).

Bi-objective column generation, Moradi *et al*. (2015)

A Branch-and-Price algorithm for the Kidney Exchange Problem, Riascos-Álvarez et al. (2020)

ACTIVIDAD COMPLEMENTARIA 1

- Lectura del artículo de Hauge et al. (2014) y control con alternativas (lunes 19 de mayo).

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A hybrid column generation approach for an industrial waste collection routing problem

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Waste collection
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Metaheuristic

ABSTRACT

This paper presents a practical roll-on/roll-off routing (RORor) problem arising in the collection of industrial waste. Skip containers, which are used for the waste collection, need to be distributed between, and collected from, a set of customers. Full containers must be driven to dump sites, while empty containers must be returned to the depot to await further assignments. Unlike, the traditional RORor problem, where vehicles may transport one skip container at a time regardless of whether it is full or not, we consider cases in which a vehicle can transport up to eight containers, at most two of which can be full. We propose a generalized set partitioning formulation of the problem and describe a hybrid column generation procedure to solve it. A fast Tabu Search heuristic is used to generate new columns. The proposed methodology is tested on nine data sets, four of which are actual, real-world problem instances. Results indicate that the hybrid column generation outperforms a purely heuristic approach in terms of both running time and solution quality. High quality solutions to problems containing up to 100 orders can be solved in approximately 15 min.

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1. Introduction

This paper focuses on a routing problem that arises in connection with the disposal of bulky waste using large containers. In a complex world where environmentally friendly solutions and recycling are on the top of the agenda, waste management systems become even more complicated. This has forced municipalities to prioritize and implement cost-effective solutions to deal with all kinds of waste. Here, we consider waste that comes from industry and which must be transported to dump sites using containers.

This particular problem belongs to the roll-on/roll-off routing (RORor) class of problems that already exist in the literature. The RORor problem is a variant of the very general framework of the Rich Vehicle Routing Problem (RVRP) – see e.g. Drexl (2012) and Schmid and Doerner (2010). As we will see, the RORor problem can be specialized further depending on the different characteristics and constraints of the problem.

We will in this paper look at a specific variant of the RORor problem in order to demonstrate how optimization-based methods and metaheuristics together can result in efficient problem solving and a flexible framework. Pratical RORor problems are today solved using very simple heuristics approaches because the constraints and characteristics make an exact approach challenging to implement and the solution time potentially intractable. In addition, all RORor problems are subtly different and therefore a framework that can be modified is necessary, and this flexibility is difficult to get with an exact approach. Companies that build software for the waste management industry therefore rely on a range of flexible heuristic framework. Finally, we also see this paper as a vehicle to push the use of optimization-based methods and metaheuristics to a wider range of vehicle routing problems than just the RORor. Many of the problems within the general definition of Rich Vehicle Routing Problems would be interesting to study using framework developed in this paper.

Simple extensions include a maximum number of trucks, many types of goods (different types of containers), capacity constraints and multiple depots. The more complex features of the problem include the introduction of disposal facilities as well as four different order types, each requiring several visits at depots, customers and/or dump sites.

Golden, Assad, and Wasil (2002) gives an introduction to waste collection as a vehicle routing problem component of the overall waste management process. A classification of RORor problems into residential, commercial and industrial is also provided. Whereas residential problems are mainly viewed as arc routing

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