

Análisis en frecuencia

Referencias

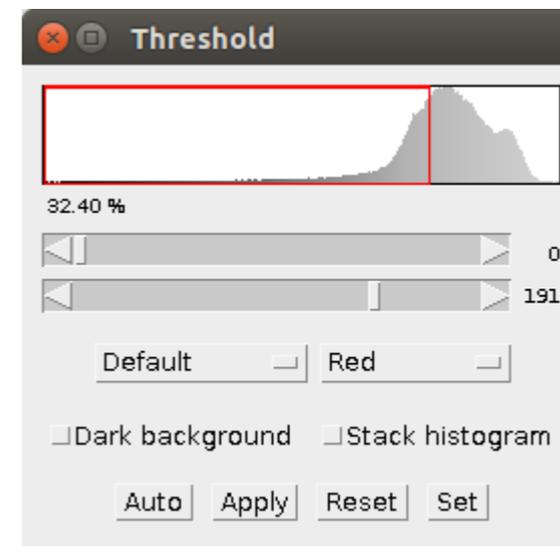
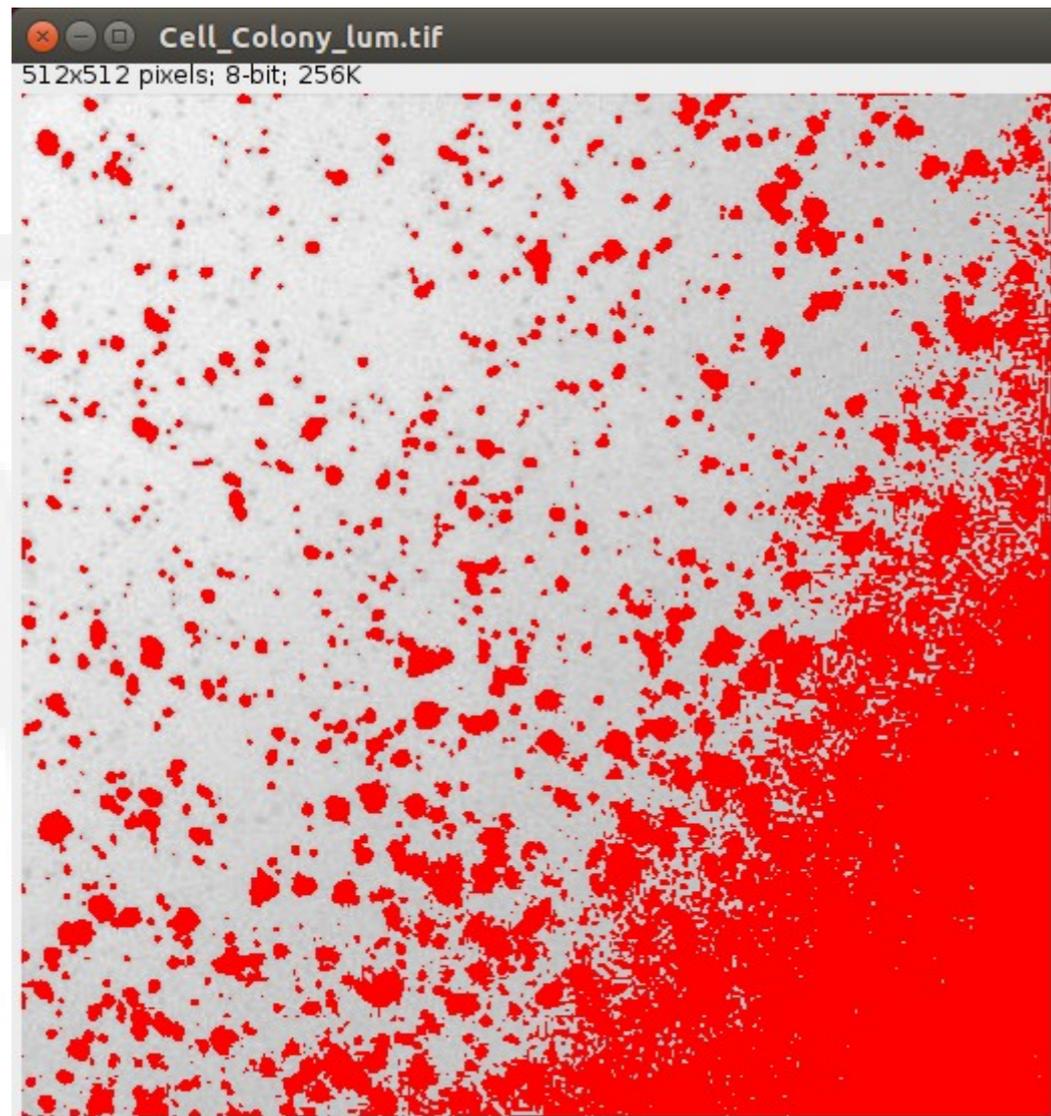
- Digital Image Processing, An Algorithmic Introduction to Java
Wilhelm Burger & Mark J. Burge. Springer, ISBN 978-1-84628-379-6
Las transparencias usan material del libro disponible en
www.imagingbook.com
- Digital Image Processing
Gonzalez & Woods
Las transparencias usan material del libro disponible en:
<http://www.imageprocessingplace.com/>

Para qué analizar en frecuencia

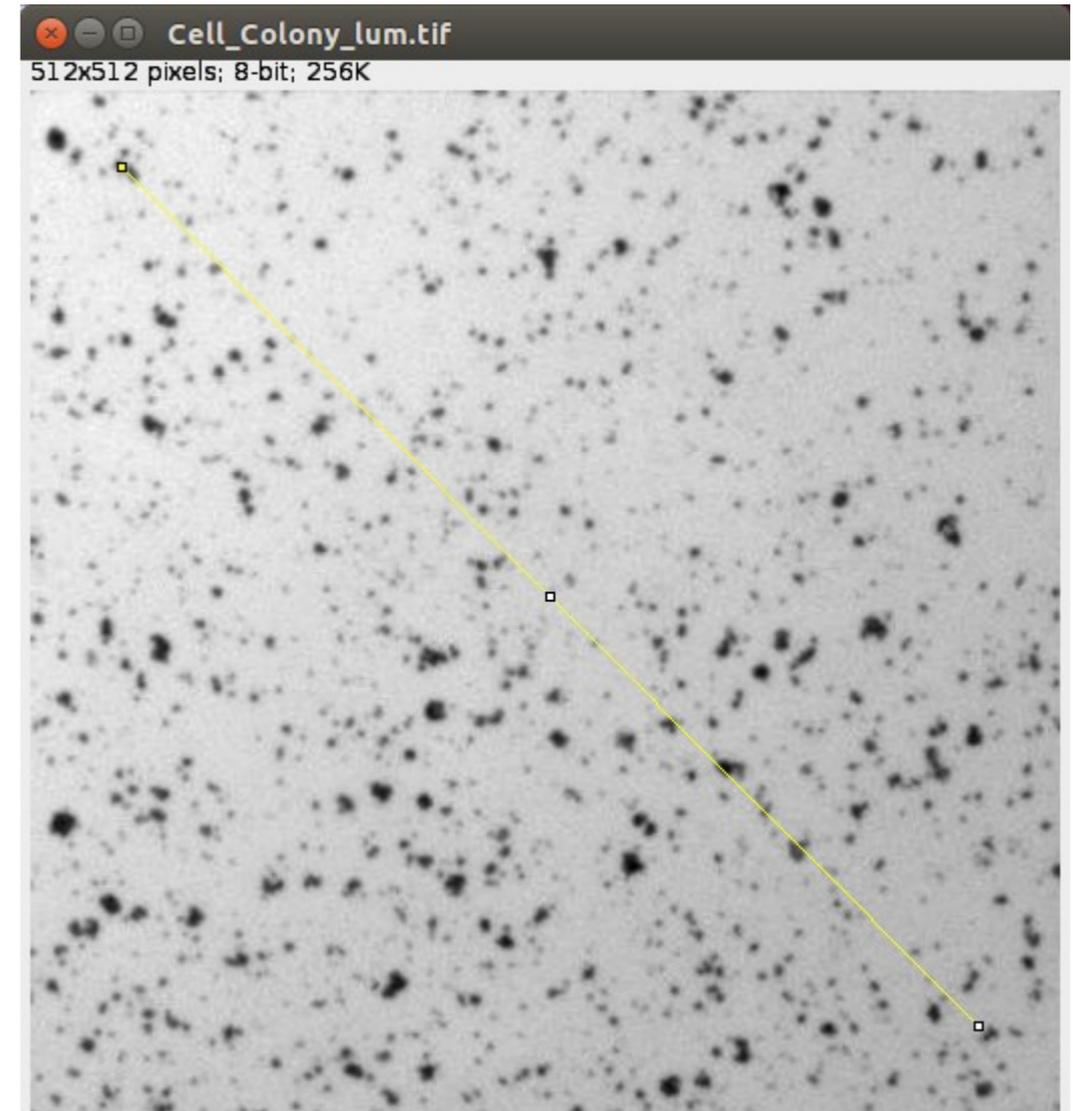
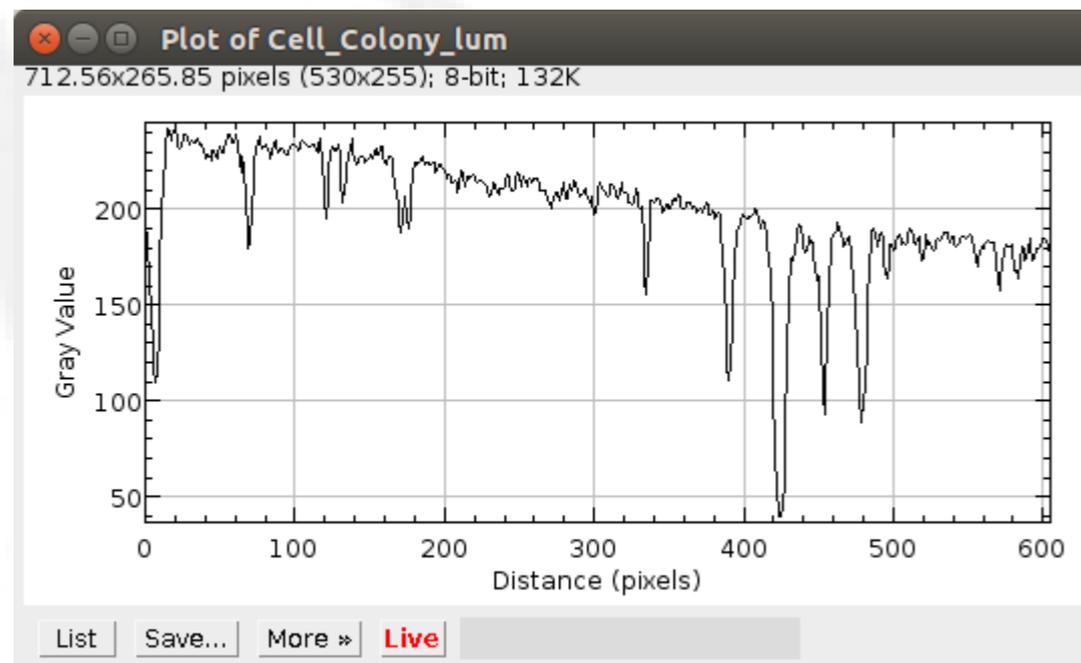
- Queremos detectar partículas en esta imagen
- Idea:
 - Umbralizar
 - Quedarnos con las partículas de cierto tamaño
- Pero ...



- No es posible encontrar un umbral global



- La imagen tiene:
 - iluminación no uniforme (variación suave)
 - ruido (variación rápida)



Para qué analizar en frecuencia

- Nos interesa en la imagen
 - Ver qué cosas varían lentamente (frecuencia baja)
 - Ver qué cosas varían rápidamente (frecuencia alta)
- Describir la imagen en función de estas componentes de distinta frecuencia
- Nos va a permitir por ejemplo filtrar componentes no deseadas.

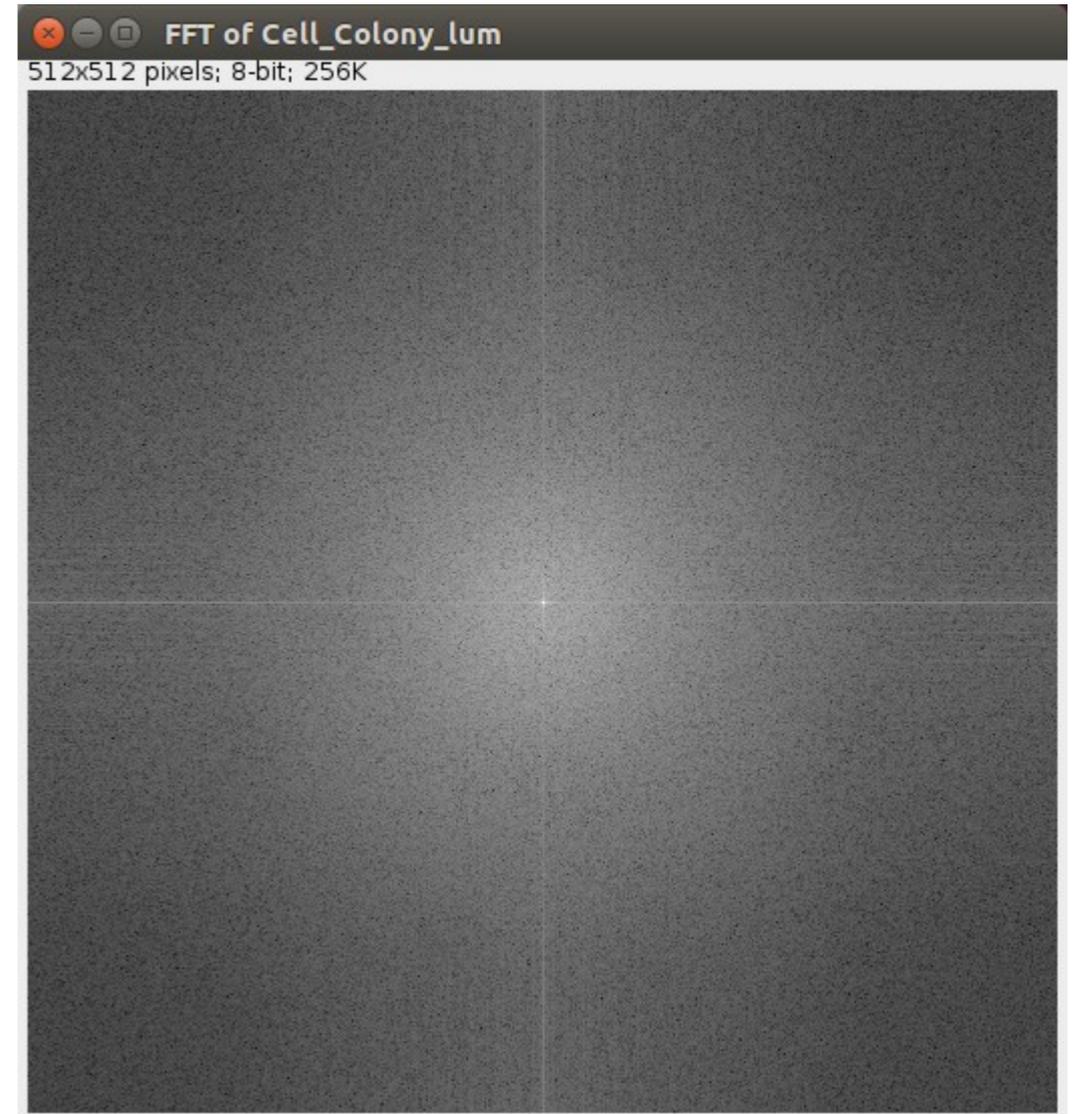


FFT



directa

inversa



Agenda

- Series de Fourier
- Transformada de Fourier
- Transformada discreta de fourier (DFT)
- Muestreo
- Transformada de Fourier 2D



Fourier

- Funciones periódicas
 - ¿Puedo representar una función periódica como suma de sinusoides?
 - ¿Qué frecuencias van a tener las sinusoides?
 - ¿Cuánto de cada senoide tengo que poner para formar mi función?

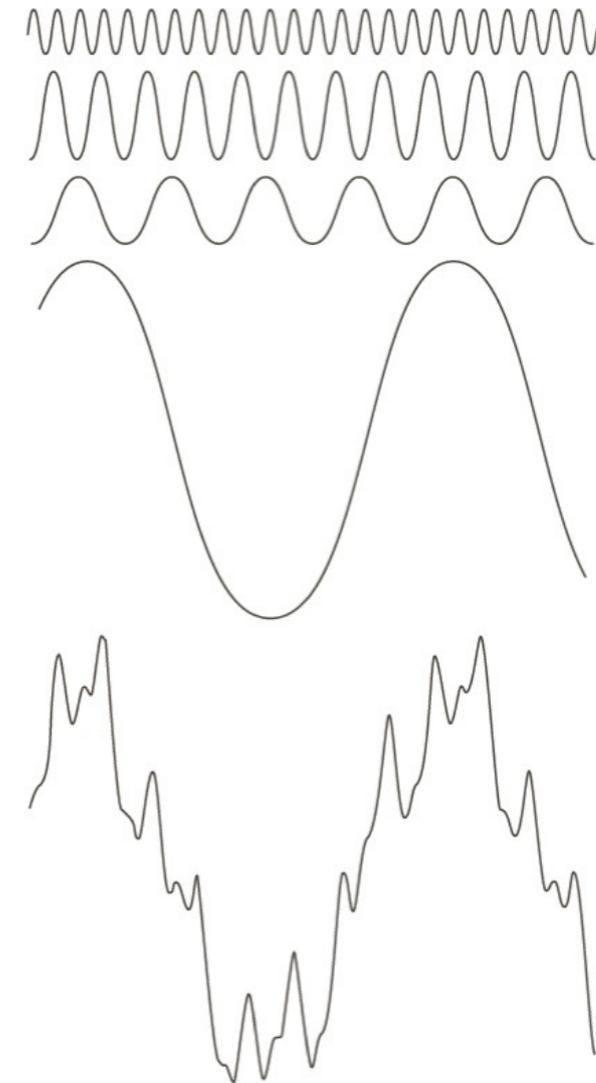


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Fourier

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Si! Con las Series de Fourier.

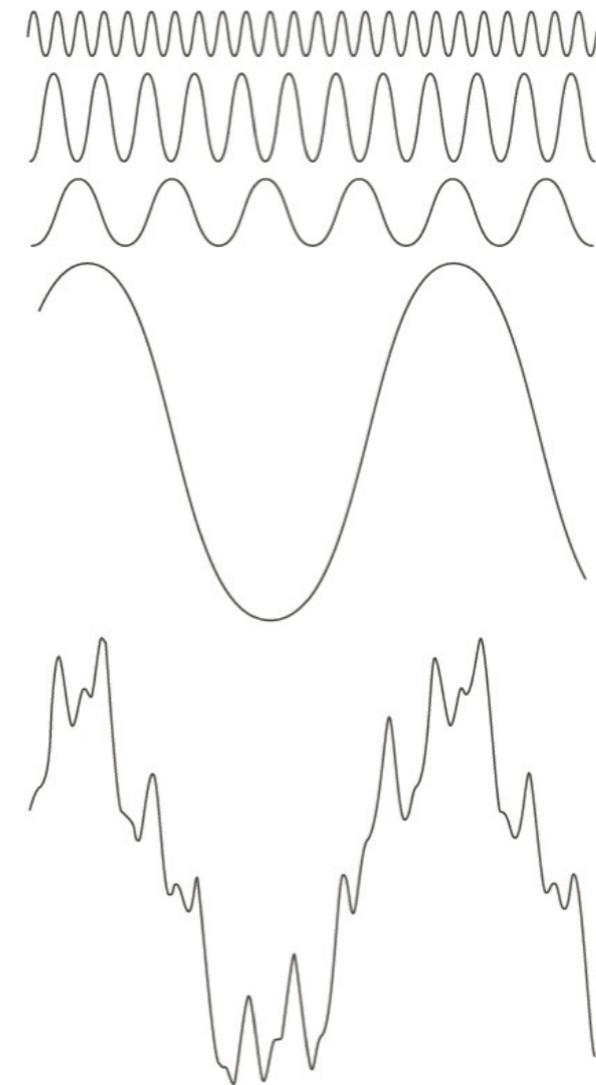


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Serie de Fourier

- Funciones periódicas
 - Podemos poner una función periódica como la suma de senos y cosenos (Serie de Fourier)

$$g(x) = \sum_{k=0}^{\infty} [A_k \cos(k\omega_0 x) + B_k \sin(k\omega_0 x)]$$

- Componentes con frecuencias múltiplos de la frecuencia de la función periódica
- ¿Cuanto valen A_k y B_k ?
 - La idea es comparar (“correlacionar”) la señal $g(x)$ con las sinusoides de manera de ver cuánto se parecen.

$$A_k = \frac{1}{T} \int_0^T g(x) \cos(k\omega_0 x) dx$$

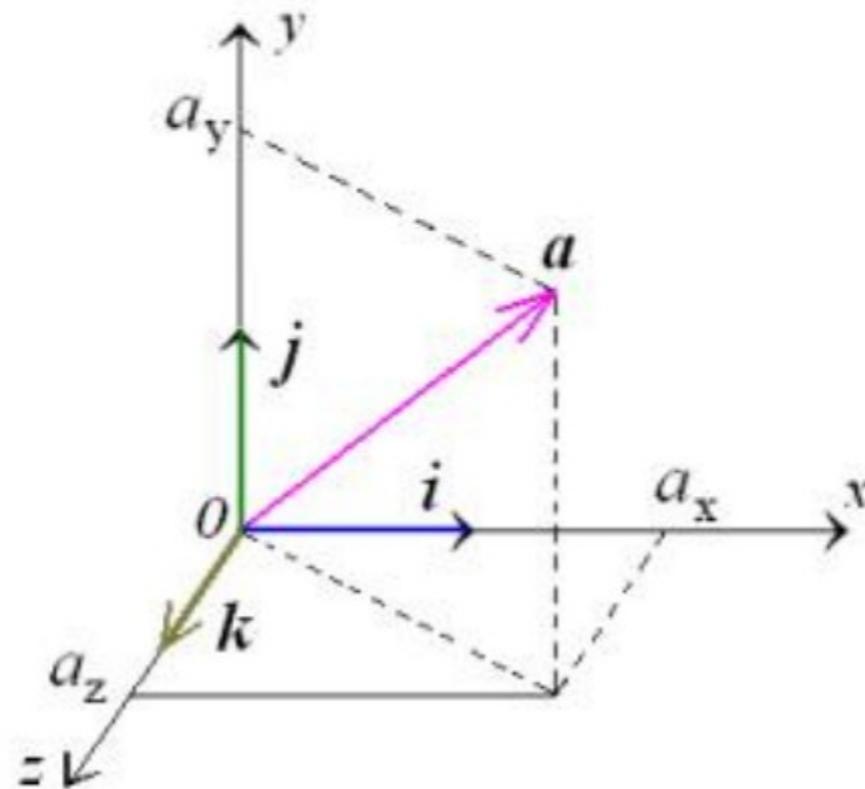
$$B_k = \frac{1}{T} \int_0^T g(x) \sin(k\omega_0 x) dx$$

$$\omega_0 = \frac{2\pi}{T}$$

Serie de Fourier

- Expresamos una función periódica como una suma ponderada de sinusoides
- Sinusoides = Funciones “base”

$$\begin{aligned}\mathbf{a} = \{a_x, a_y, a_z\} &= a_x \{1, 0, 0\} + a_y \{0, 1, 0\} + a_z \{0, 0, 1\} \\ &= a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}\end{aligned}$$



Serie de Fourier

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 x) + b_k \sin(k\omega_0 x)$$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{x_0} \quad \text{frez ang fundamental}$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \underbrace{f_k(x)}_{\text{armonico de } f}$$



$$\int_0^{x_0} f(x) \cos(m\omega_0 x) dx =$$

$$\int_0^{x_0} a_0 \cos(m\omega_0 x) dx + \int_0^{x_0} \sum a_k \cos(k\omega_0 x) \cos(m\omega_0 x) dx$$

$$+ \int_0^{x_0} \sum b_k \sin(k\omega_0 x) \cos(m\omega_0 x) dx$$

$$= \begin{cases} a_0 x_0 & \text{si } m=0 \\ \frac{a_k x_0}{2} & \text{si } m \neq 0 \end{cases} \Rightarrow a_0 = \frac{1}{x_0} \int_0^{x_0} f(x) dx$$
$$\Rightarrow a = \frac{2}{x_0} \int_0^{x_0} f(x) \cos(k\omega_0 x) dx$$

tro termo multiplicando por $\sin(k\omega_0 x)$

$$a_0 = \frac{1}{X_0} \int_0^{X_0} f(x) dx$$

$$a_k = \frac{2}{X_0} \int_0^{X_0} f(x) \cos(k\omega_0 x) dx$$

$$b_k = \frac{2}{X_0} \int_0^{X_0} f(x) \sin(k\omega_0 x) dx$$

Forma 'com fase'

$$r \cos(A + \phi) = p \cos(A) + q \sin(A)$$

$$\cos(A + \phi) = \frac{p}{r} \cos(A) + \frac{q}{r} \sin(A)$$

$$\cos(A + \phi) = \cos \phi \cos(A) - \sin \phi \sin(A)$$

$$\frac{p}{r} = \cos \phi \quad \frac{q}{r} = -\sin \phi \quad \operatorname{tg} \phi = -\frac{q}{p}$$

$$\cos^2 \phi + \sin^2 \phi = \frac{p^2}{r^2} + \frac{q^2}{r^2} = 1$$

$$\Rightarrow r = \sqrt{p^2 + q^2} \quad \phi = \operatorname{arctg}\left(-\frac{q}{p}\right)$$

$$f(x) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 x + \phi_k)$$

$$A_0 = a_0 \quad A_k = \sqrt{a_k^2 + b_k^2} \quad \phi_k = \operatorname{arctg}\left(-\frac{b_k}{a_k}\right)$$

Forma compleja

$$\cos(x) = \frac{1}{2} (e^{jx} + e^{-jx})$$

$$\sin(x) = \frac{1}{2j} (e^{jx} - e^{-jx})$$

$$\begin{aligned} f_k(x) &= a_k \cos(k\omega_0 x) + b_k \sin(k\omega_0 x) \\ &= \frac{a_k}{2} (e^{jk\omega_0 x} + e^{-jk\omega_0 x}) + \frac{b_k}{2j} (e^{jk\omega_0 x} - e^{-jk\omega_0 x}) \\ &= \underbrace{\frac{(a_k - jb_k)}{2}}_{C_k} e^{jk\omega_0 x} + \underbrace{\frac{(a_k + jb_k)}{2}}_{C_{-k}} e^{-jk\omega_0 x} \end{aligned}$$

$$k > 0$$

$$f(x) = \sum_{k=-\infty}^{+\infty} C_k e^{jk\omega_0 x}$$

$$C_0 = a_0$$

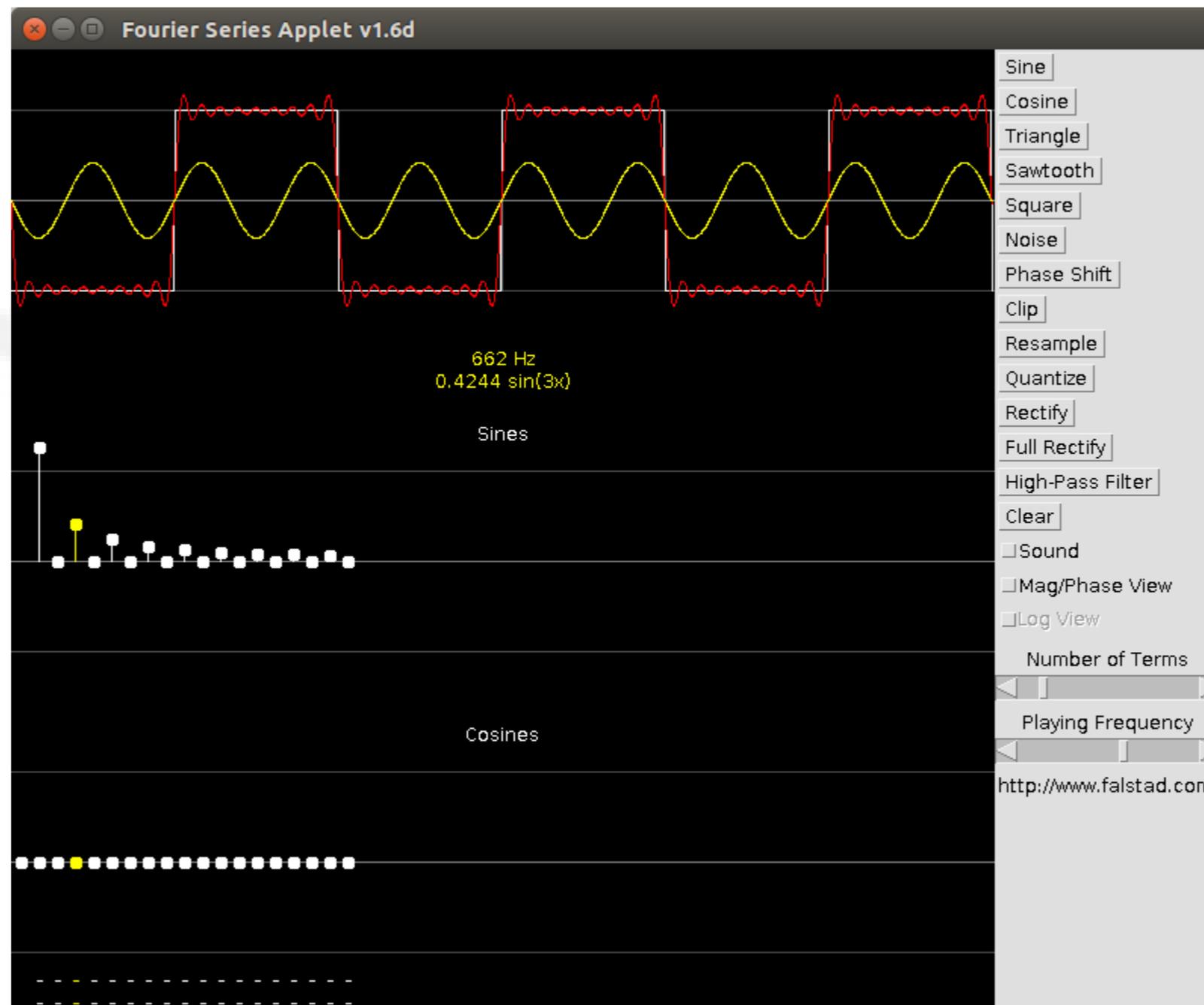
$$C_{-k} = C_k^*$$

los términos en k positivos
y negativos son complejos
conjugados \Rightarrow reconstruimos
señal real

$$C_k = \frac{1}{x_0} \int_0^{x_0} f(x) e^{-jk\omega_0 x} dx$$

Serie de Fourier

- <http://www.falstad.com/fourier/>



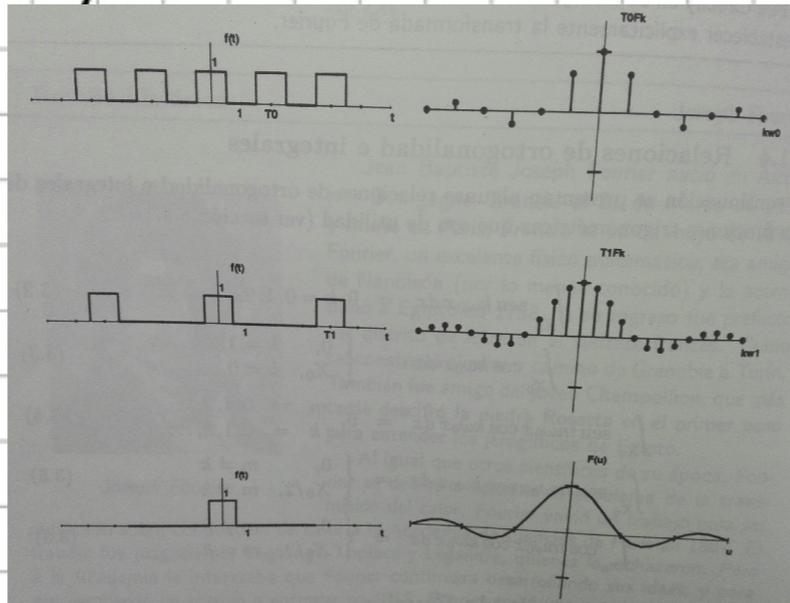
Fourier

- ¿Se puede extender la idea a funciones no periódicas ?



Transformada de Fourier

Se puede ver como el límite de la serie



$$F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(u) e^{j2\pi ux} dx$$

Serie

$$f(x) = \sum C_k e^{j k \omega_0 x}$$

$$C_k = \frac{1}{X_0} \int_0^{X_0} f(x) e^{-j k \omega_0 x} dx$$

- $f(x)$ de período X_0
- Componentes de freq C_k concentrados en $k \omega_0$

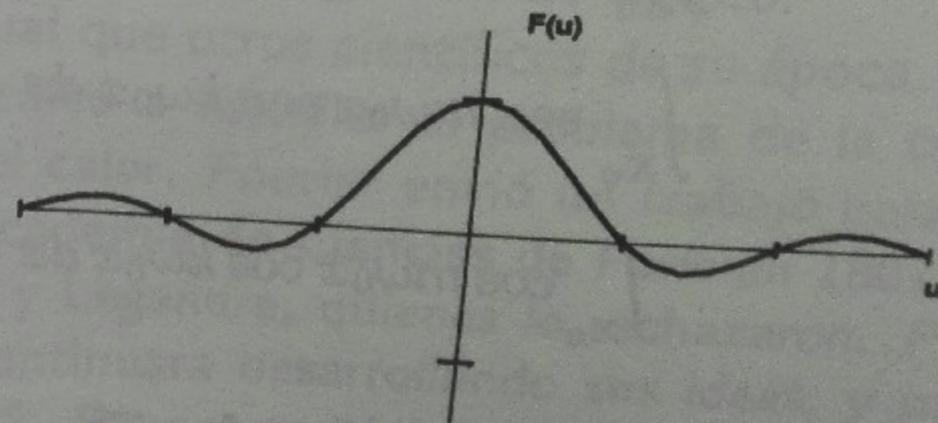
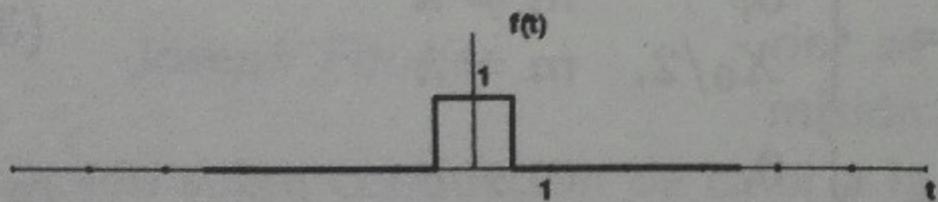
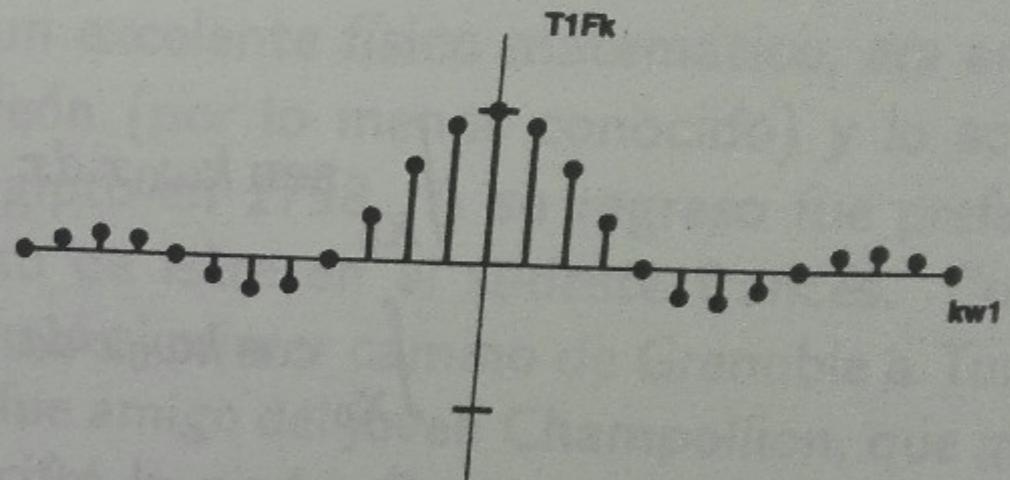
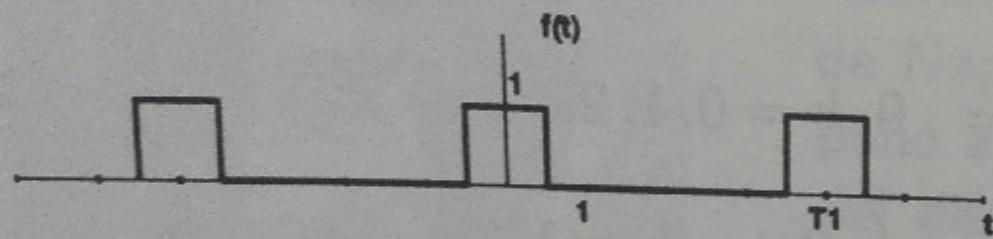
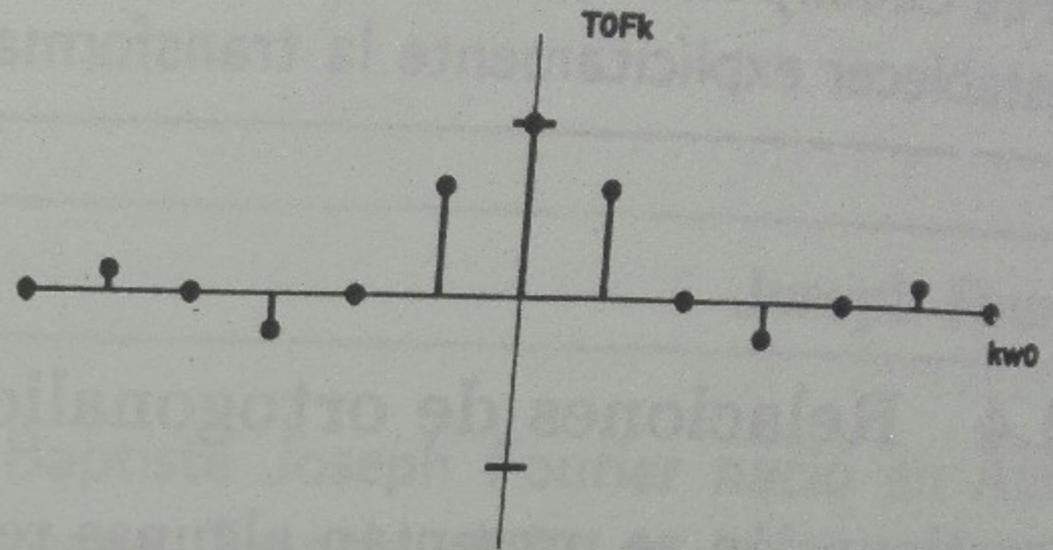
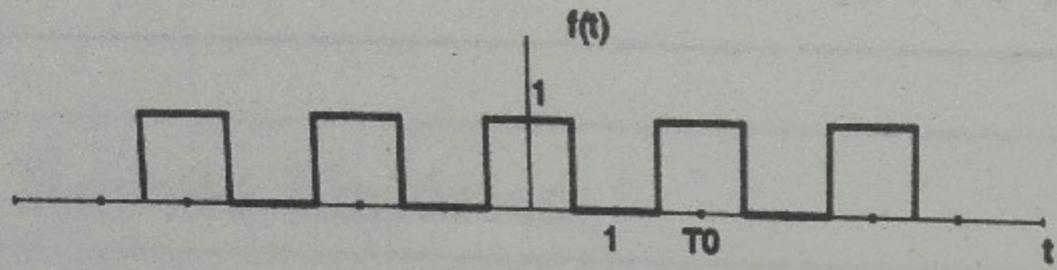
$$\text{con } \omega_0 = \frac{2\pi}{X_0}$$

Transf

$$f(x) = \int_{-\infty}^{+\infty} F(u) e^{j2\pi ux} dx$$

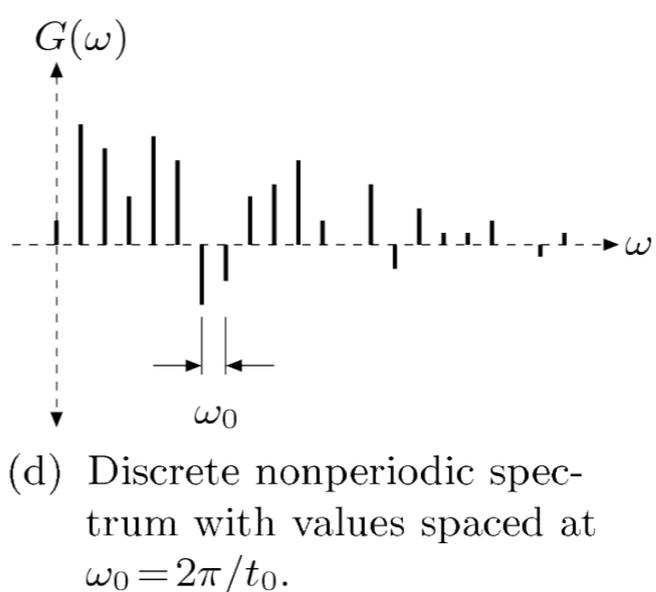
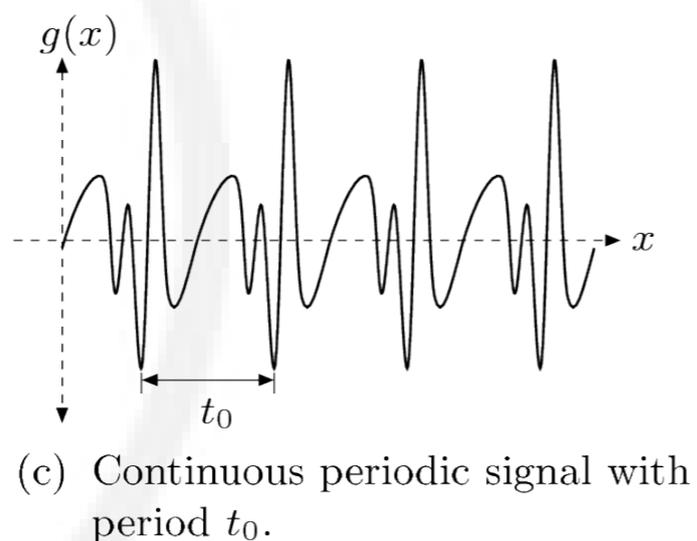
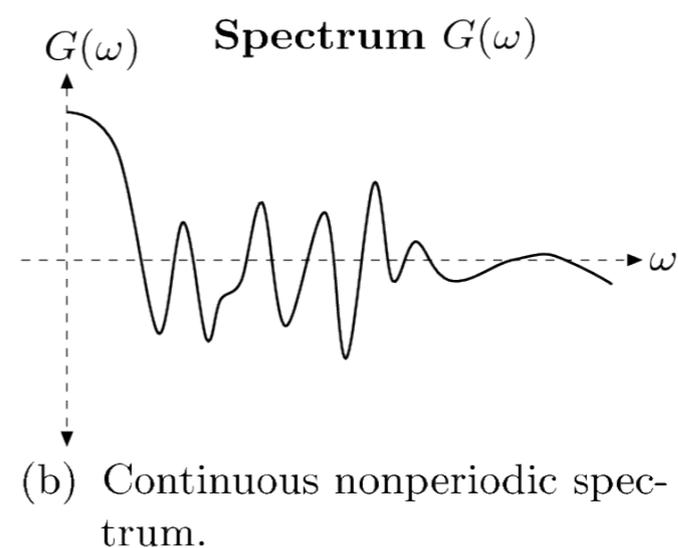
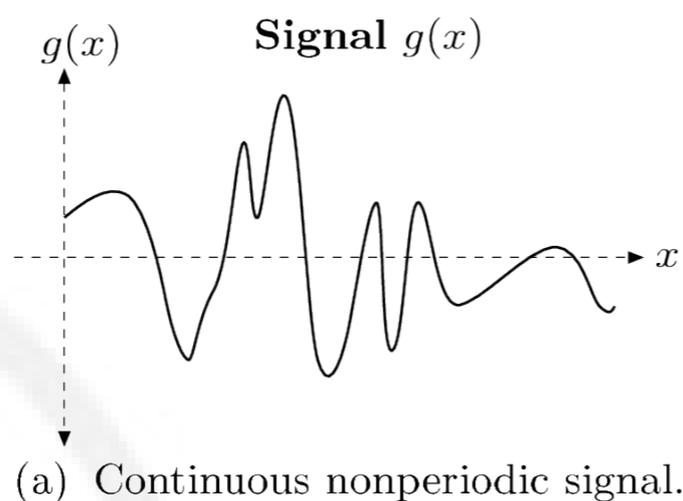
$$F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} dx$$

- $f(x)$ no periódica
- Componentes de frecuencia continuas



Transformada de fourier

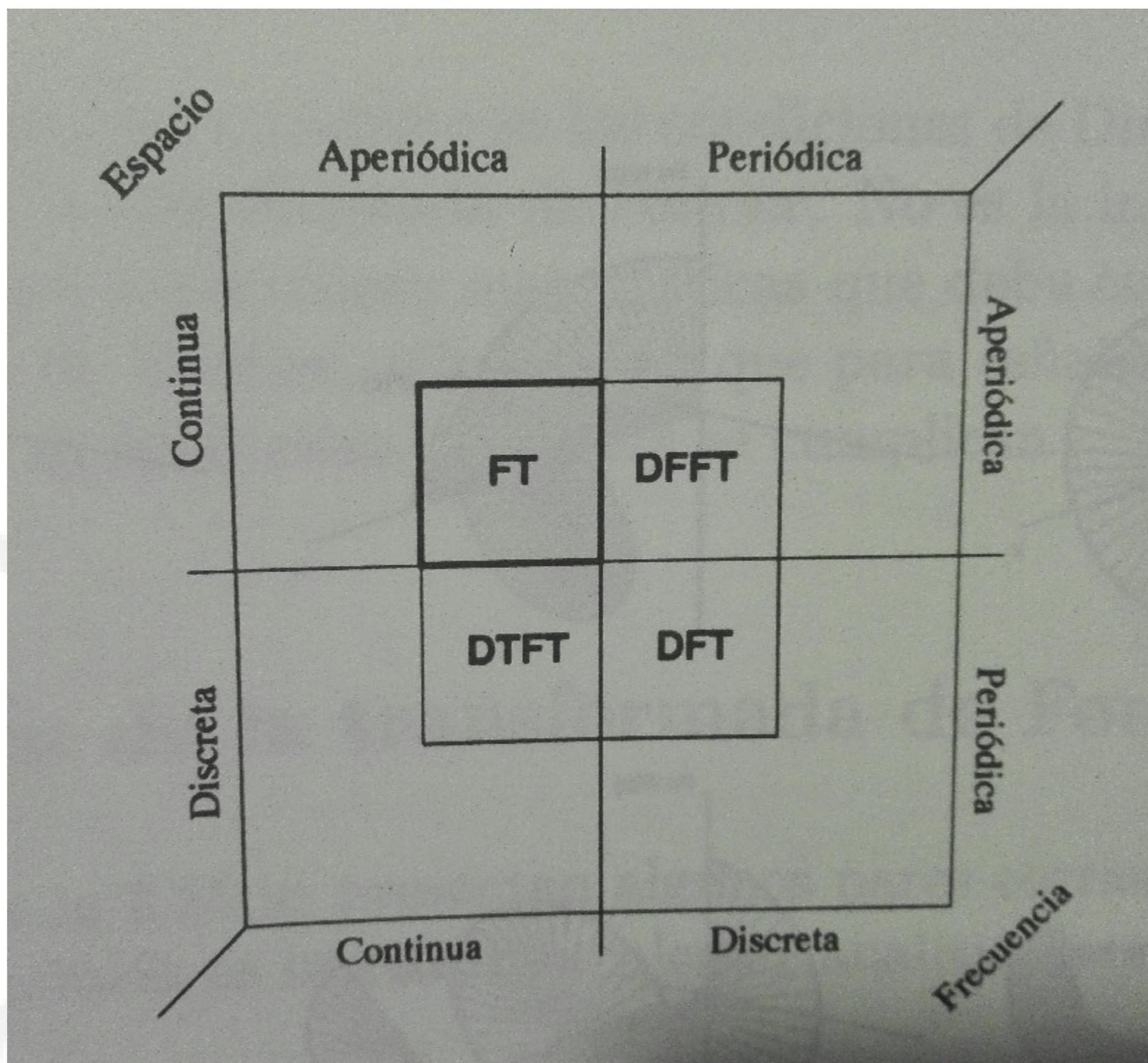
- ¿Se puede extender la idea a funciones no periódicas ?
 - Si, pero ahora en lugar de tener frecuencias discretas tenemos un continuo



Transformada de Fourier

- Qué quiere decir **transformar** la señal?
 - Transformamos la señal para verla / analizarla de otra manera
 - Expresamos la señal en términos de sus componentes de frecuencia
 - Podemos movernos en los dos sentidos sin perder información
 - Transformada directa: del espacio a frecuencia
 - Transformada inversa: de frecuencia al espacio

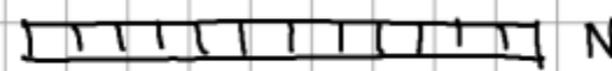
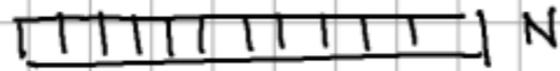




DFT Discrete Fourier Transform

$f[m]$ discreta

$F[k]$ discreta



$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j \frac{2\pi k n}{N}}$$

$$f[m] = \sum_{k=0}^{N-1} F[k] e^{j \frac{2\pi k m}{N}}$$

En general para nosotros

$f[m]$ serán las muestras de una señal real

$\Rightarrow F[k]$ será un array de números complejos

FFT = DFT calculada rápidamente

DFT - $O(N^2)$

FFT $O(N \log_2 N)$

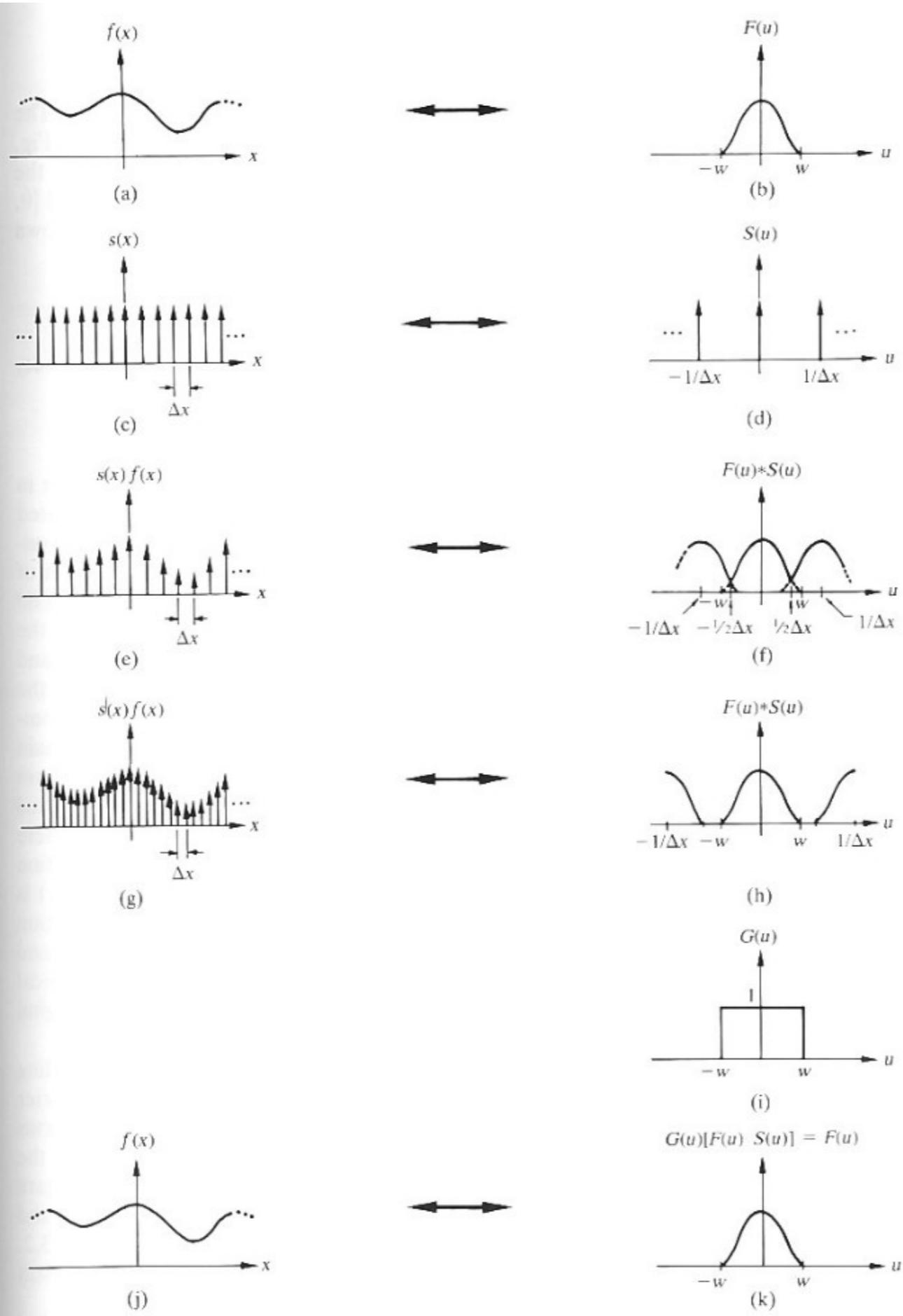
$N = 1024$

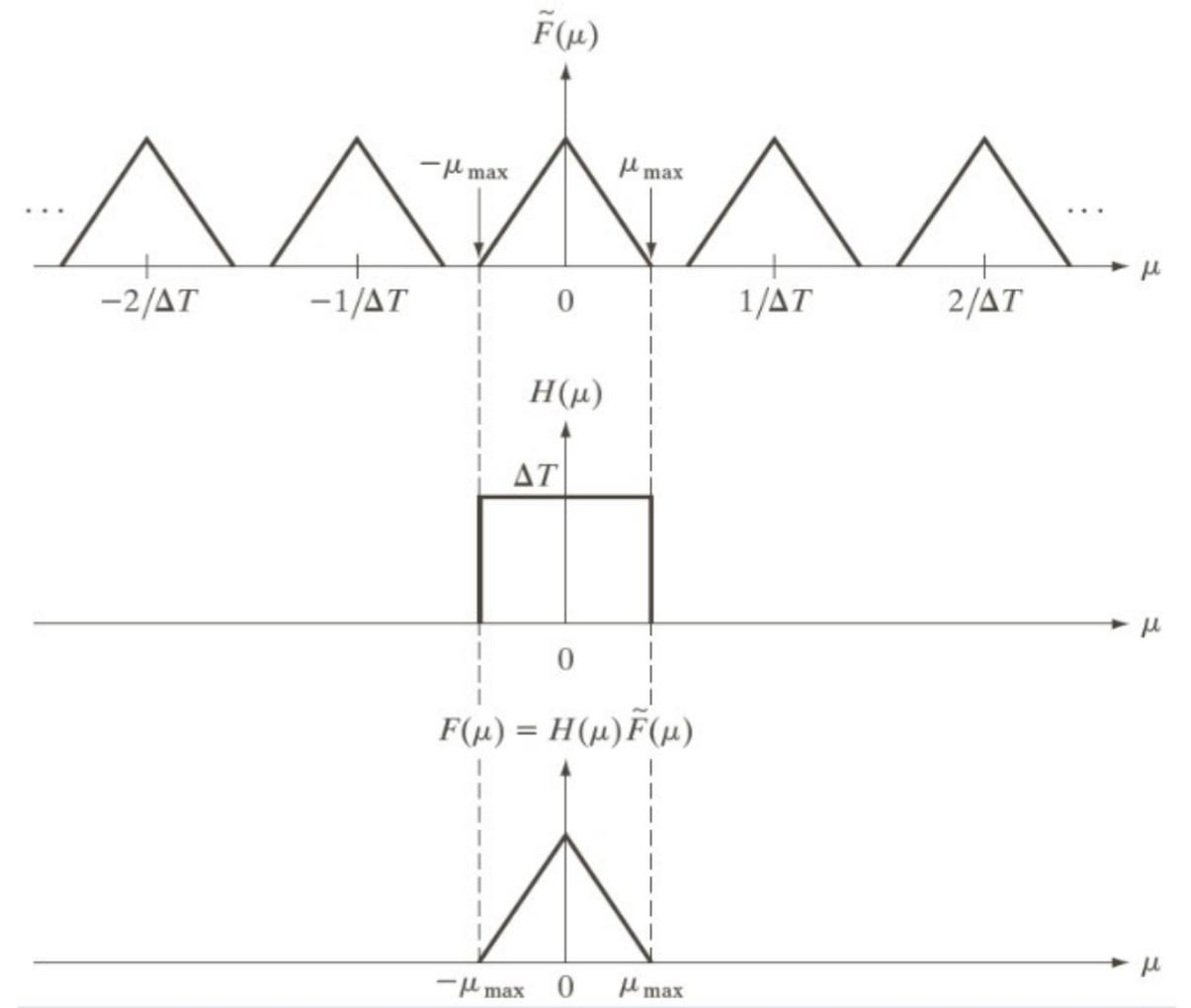
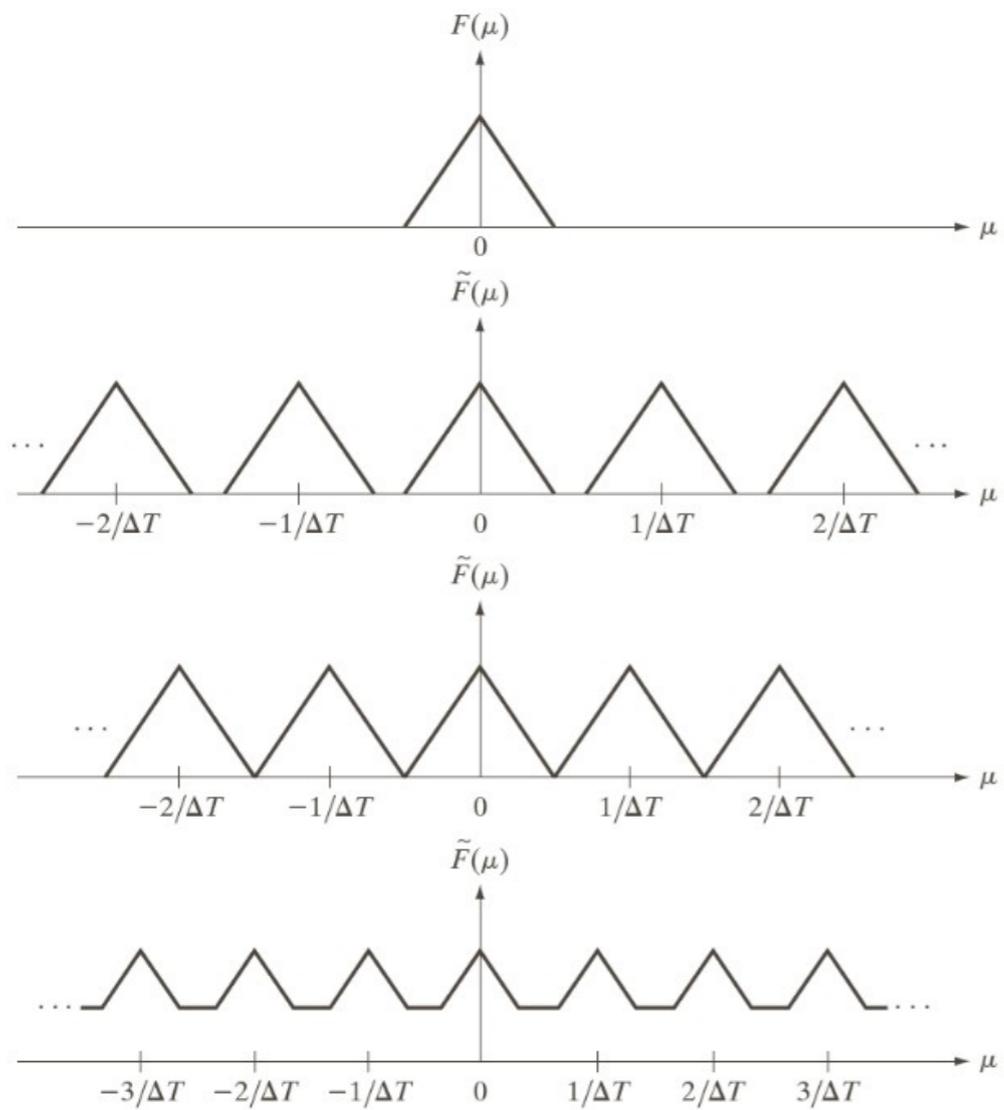
$N = 1024$

$N^2 = 1048576$

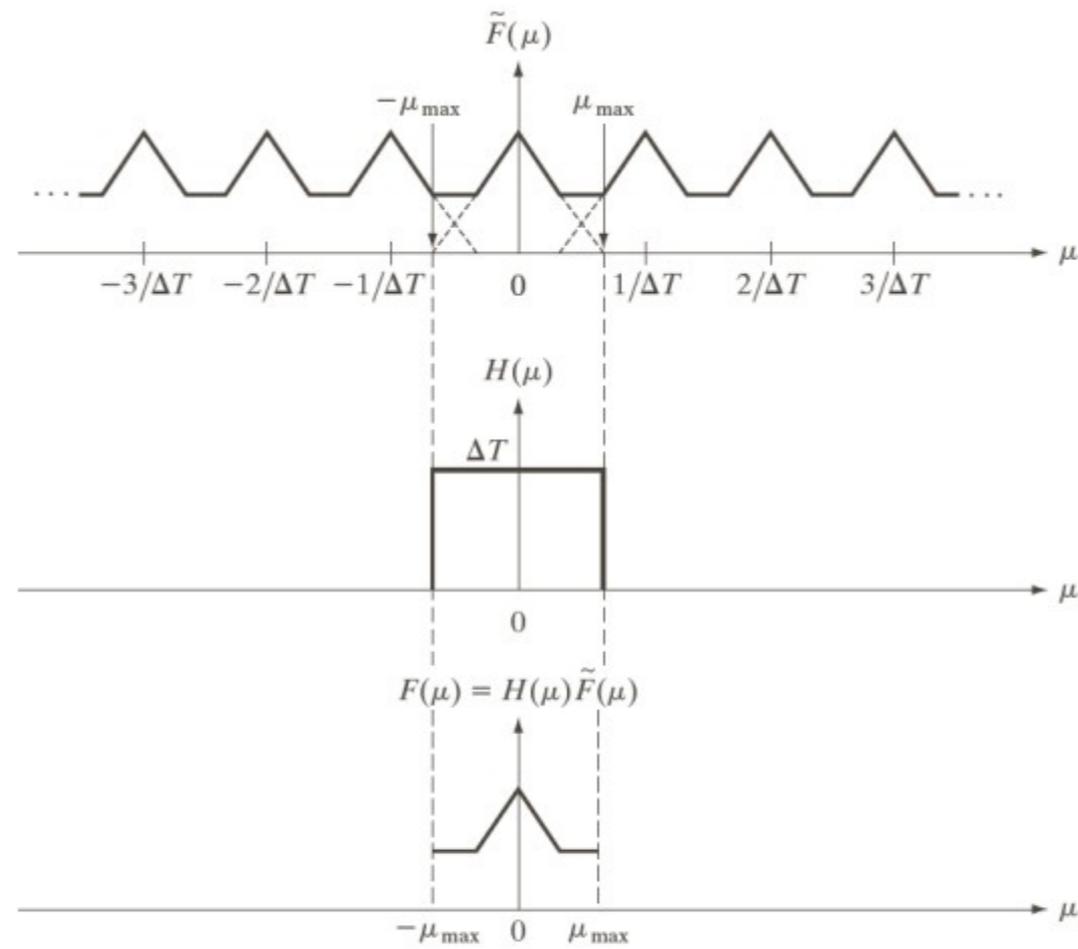
$N \log_2 N = 10240$

Muestreo

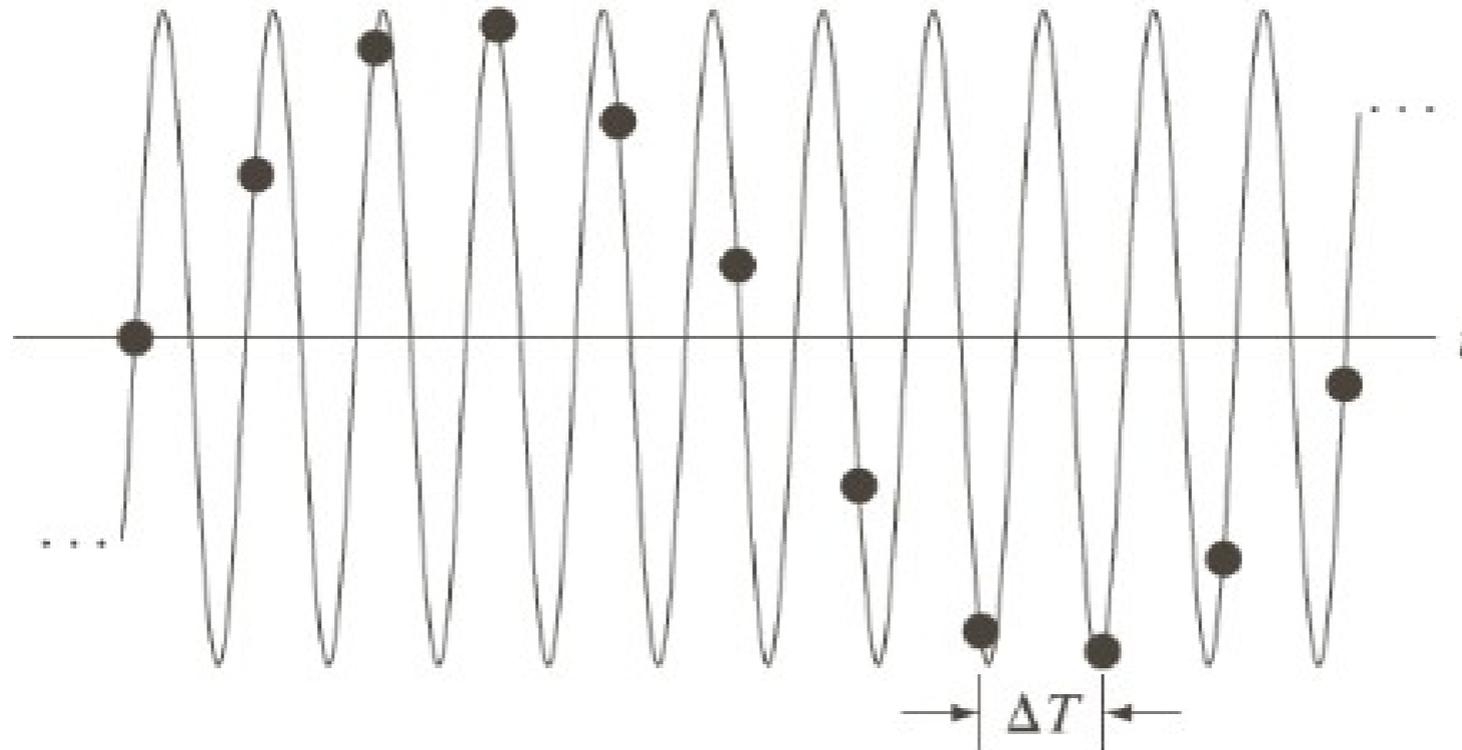




Aliasing



a
b
c

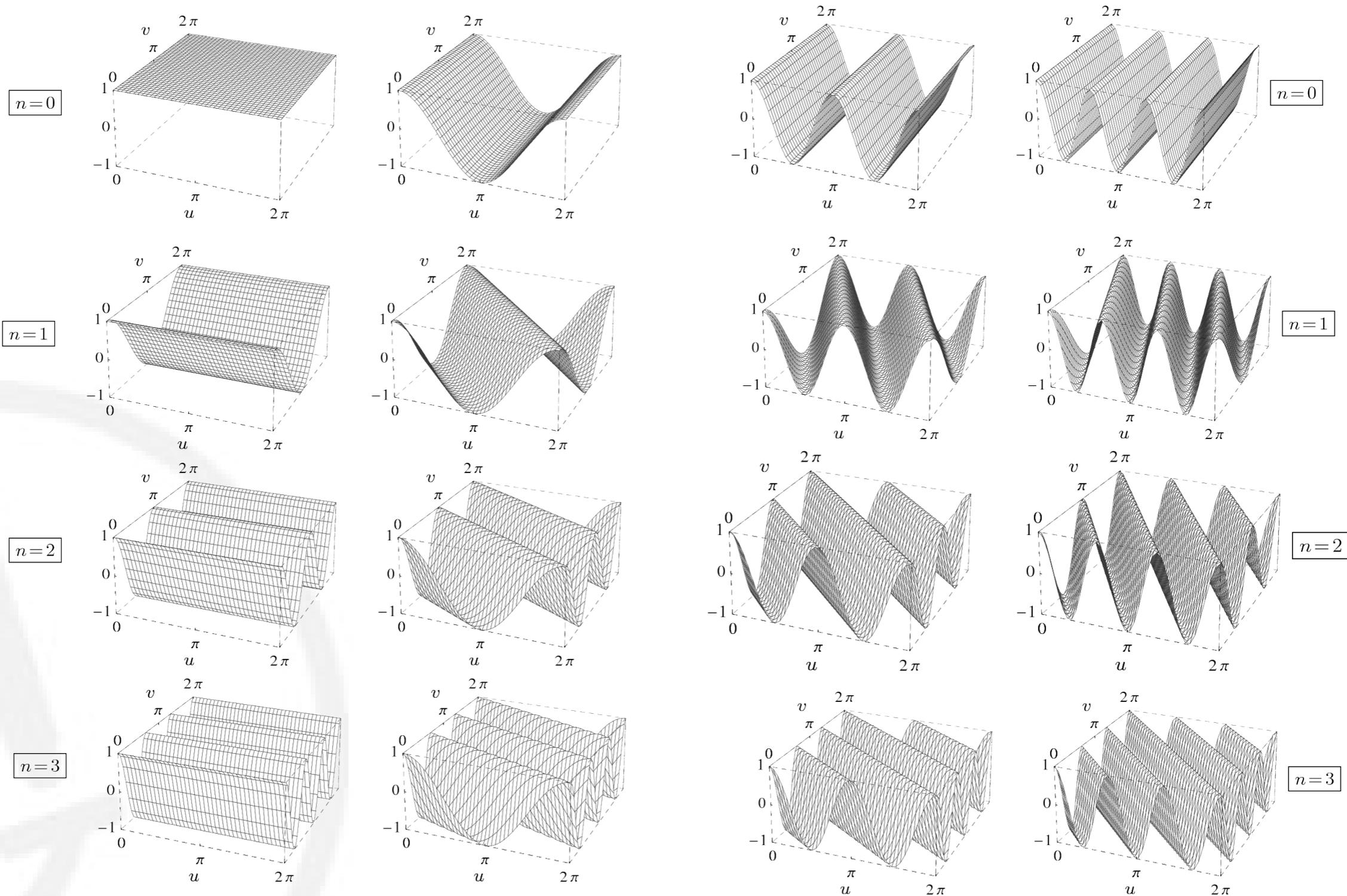


Transformada de Fourier 2D

- Ahora nuestras funciones base son sinusoides en 2D
- Vamos a comparar nuestra señal contra sinusoides espaciales con distintas frecuencias y orientaciones



Transformada de Fourier 2D

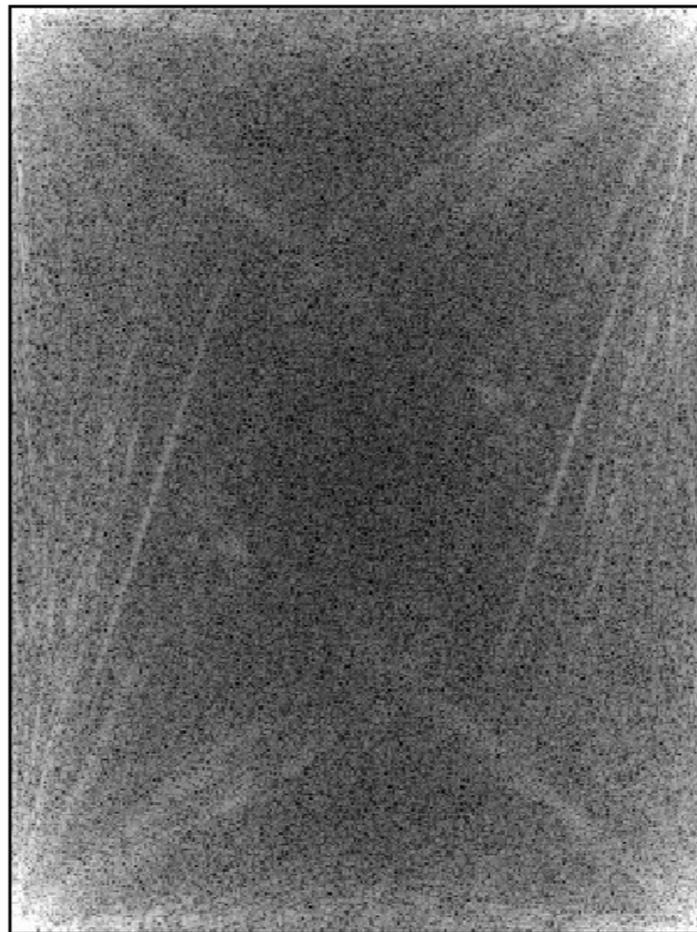


Transformada de Fourier 2D

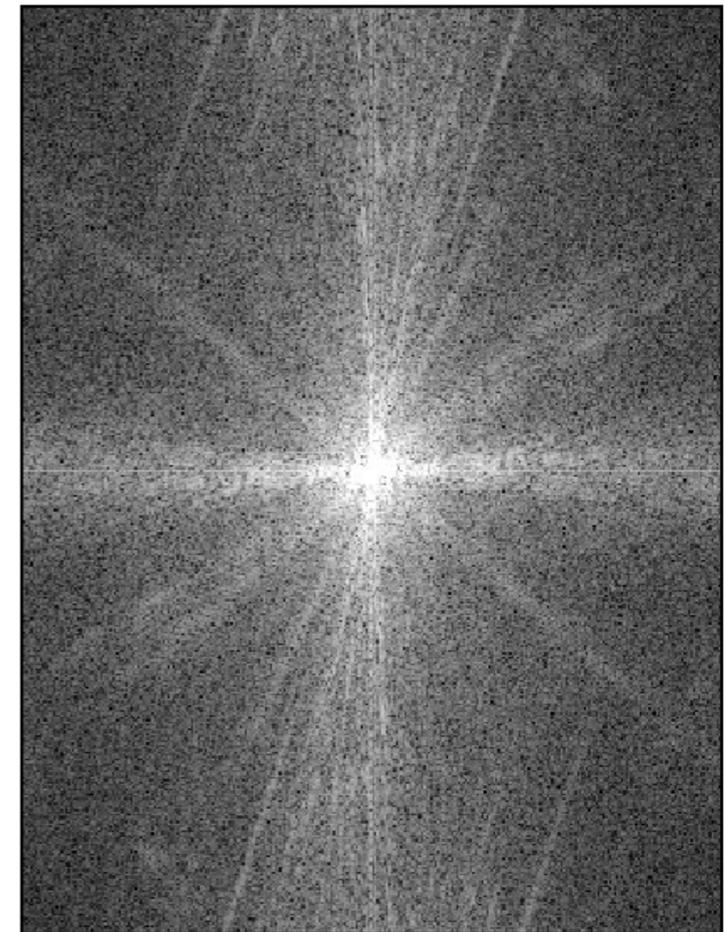
- Visualización de la transformada: espectro



(a)



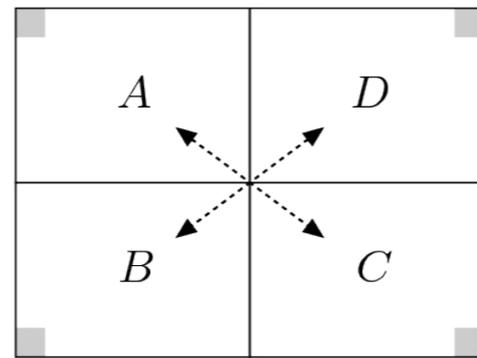
(b)



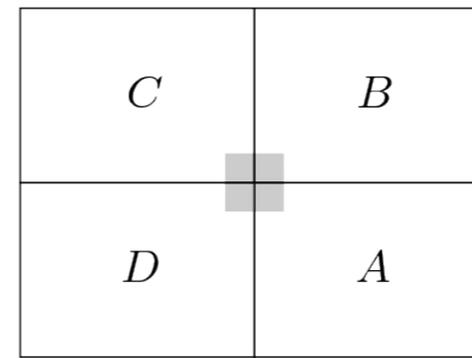
(c)

Transformada de Fourier 2D

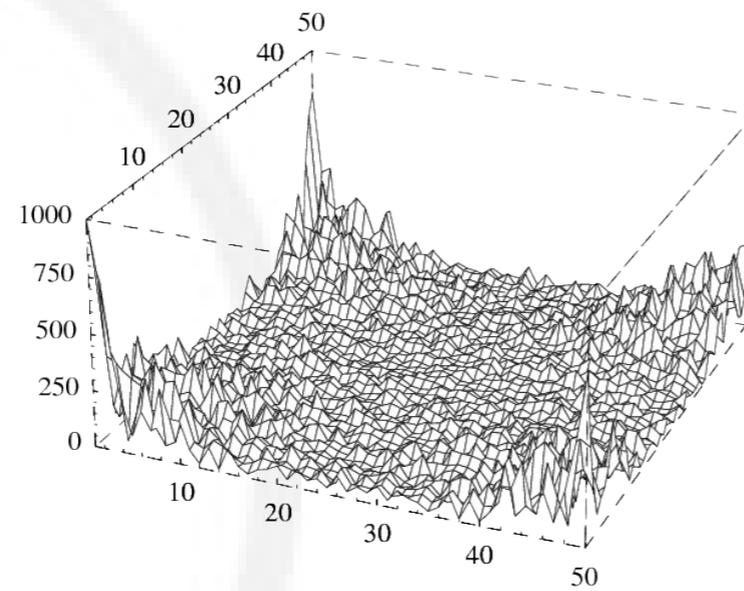
- Visualización de la transformada



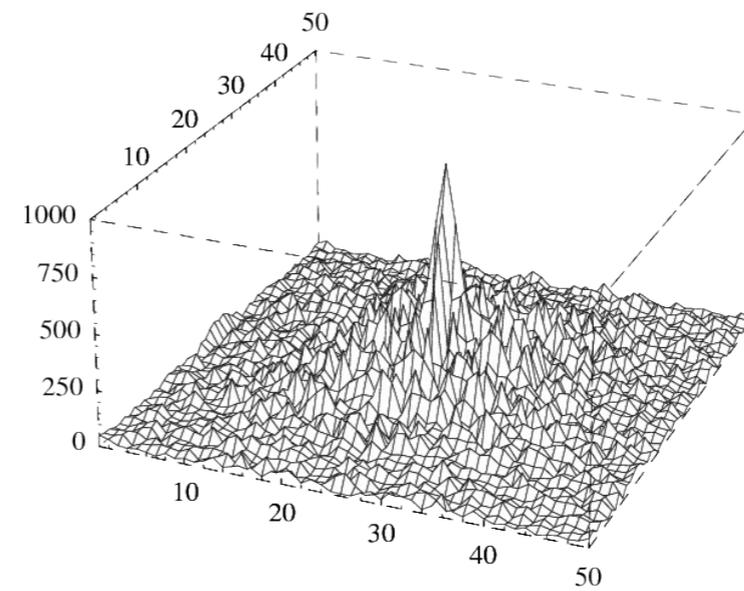
(a)



(b)



(c)



(d)

Transformada de Fourier 2D - Propiedades



Pruebas en Fiji

Período - Frecuencia

a) Crear una imagen sinusoidal

a.1) Crear una nueva imagen de 512x512 de 32 bits. [File → New → Image]

a.1) Llenar la imagen con una senoide

Process → Math → Macro... →

Code:

b) Ver el período en pixels de la senoide sobre la horizontal y en la vertical

b.1) Trazar una línea

b.2) Graficar el perfil de esa línea (Analyze → Plot profile)

b.3) Ver el período en pixels en la horizontal y vertical (se puede hacer interactivo seteando el modo "Live" en el gráfico del perfil)

Pregunta 1: ¿Qué son A, C y T ?

c) Ver la transformada de Fourier (Process → FFT → FFT)

c.1) Ver la ubicación de los principales picos en la transformada (distancia al centro, orientación con respecto a la senoide)

d) Repetir lo anterior para otras senoideas:

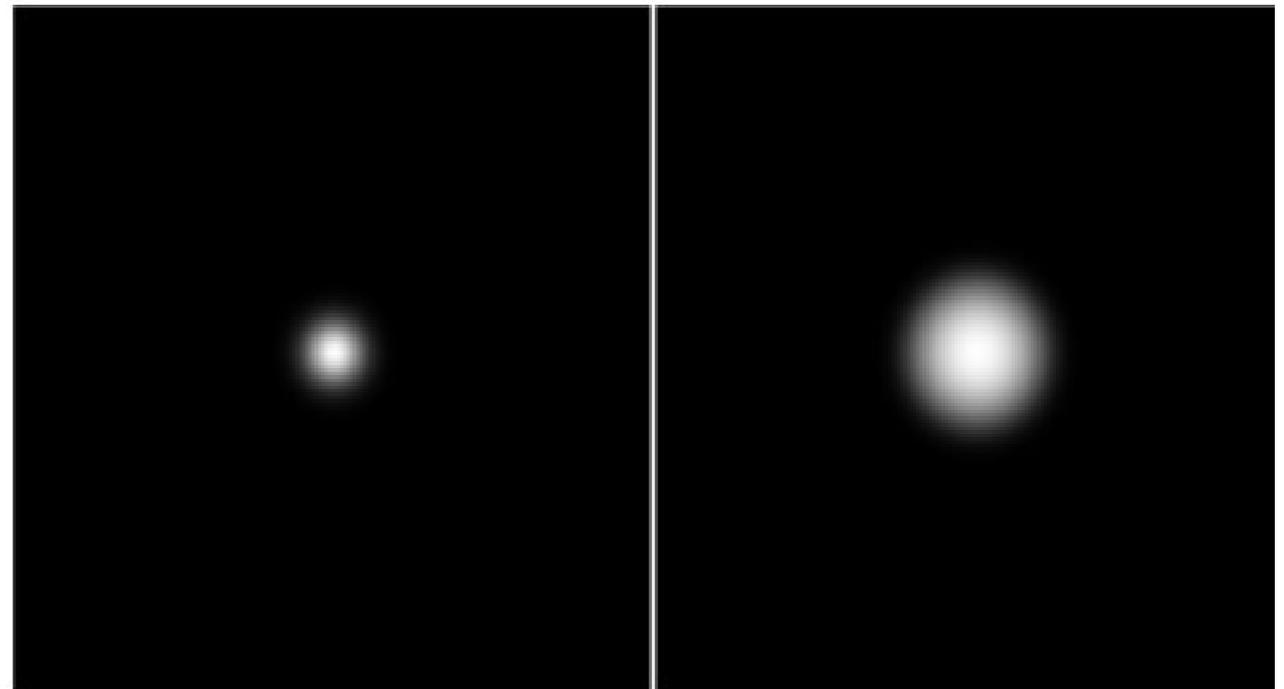
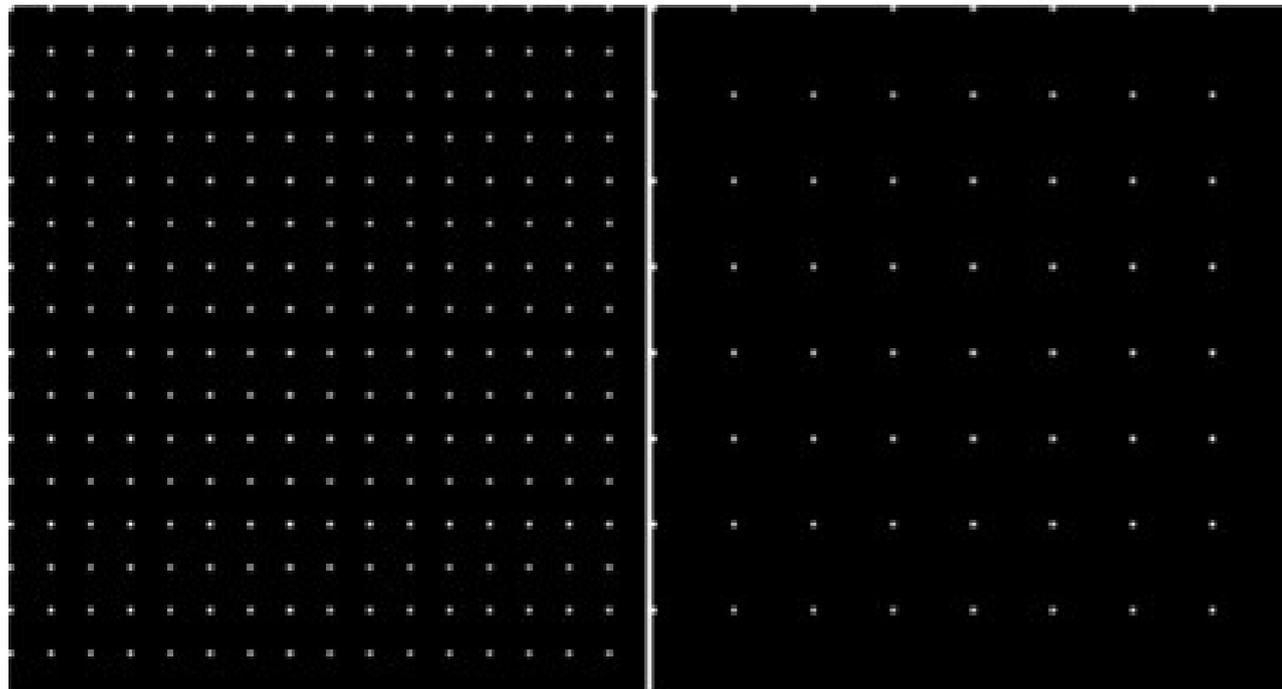
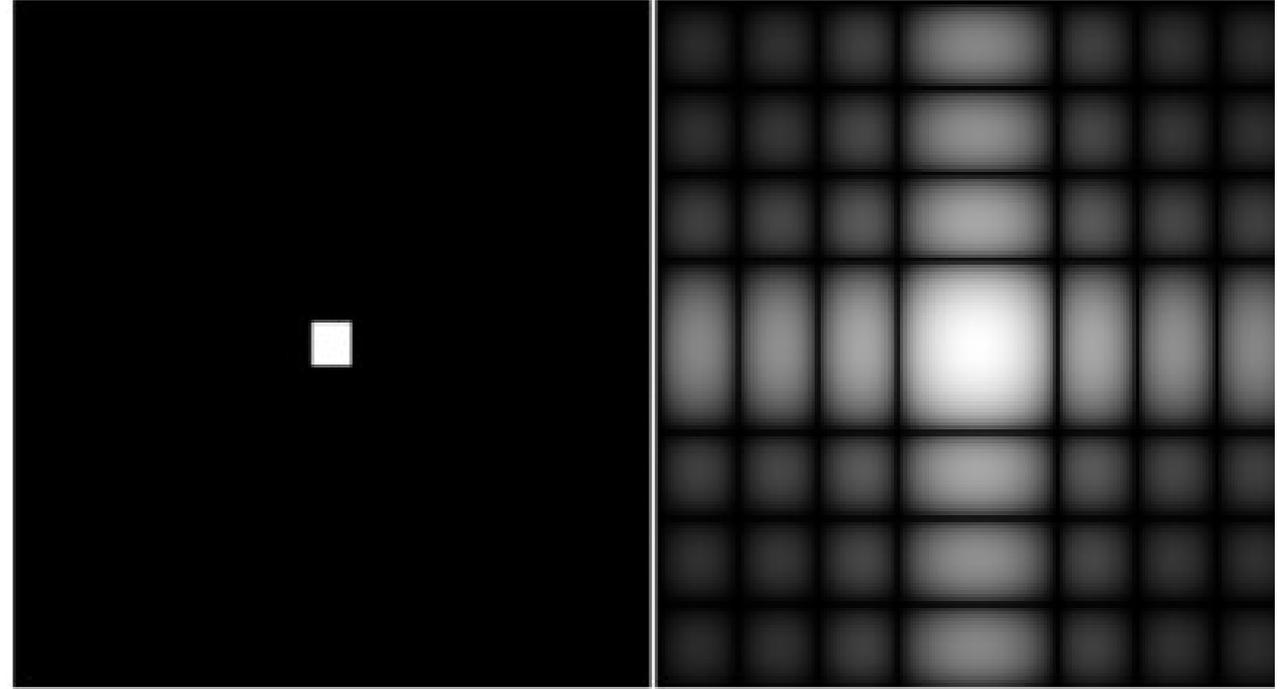
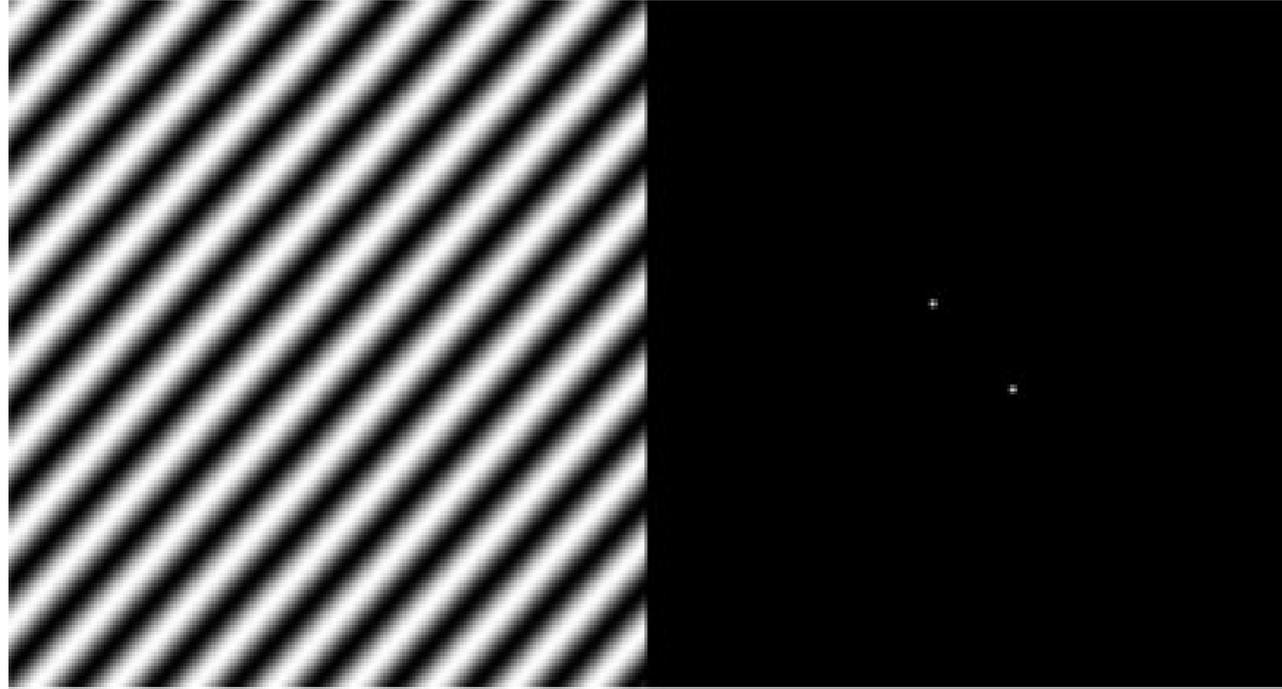
d.1) Code:

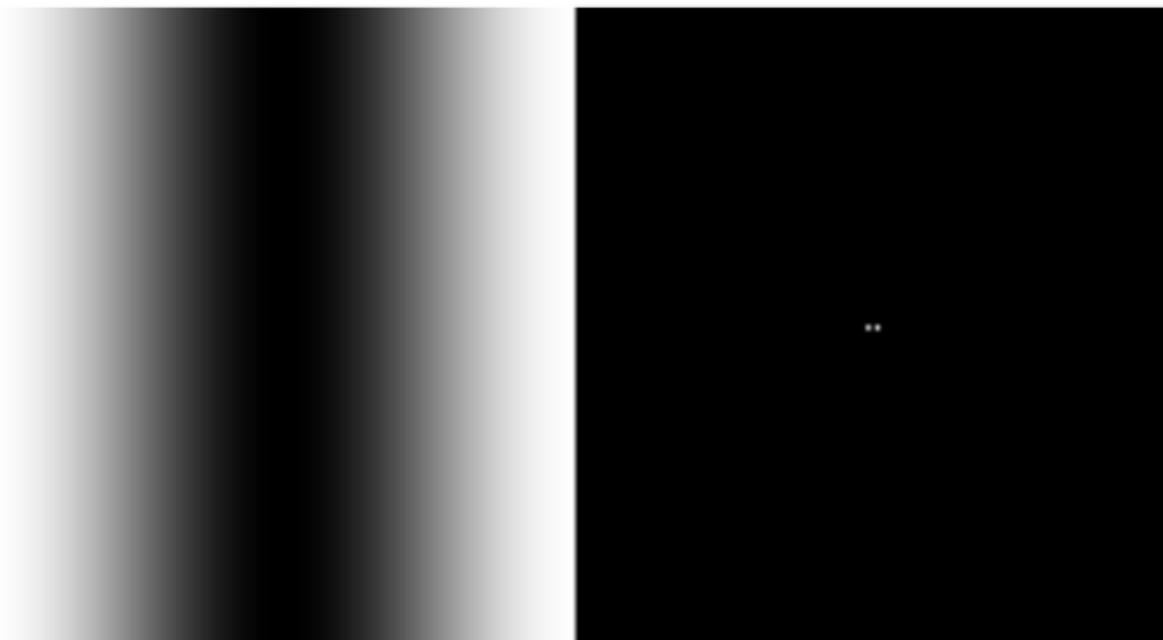
d.2) Code:

d.3) Code:

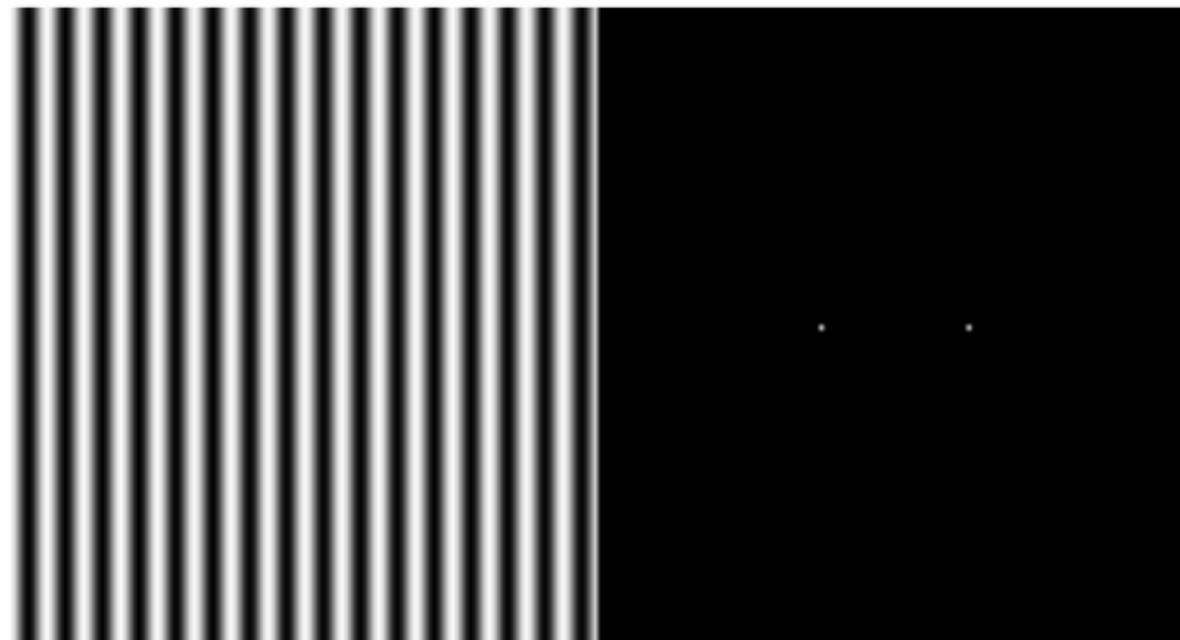
d.4) Code:

Pregunta 2: ¿Cómo se relaciona la ubicación de los picos con los valores de A, C y T ?

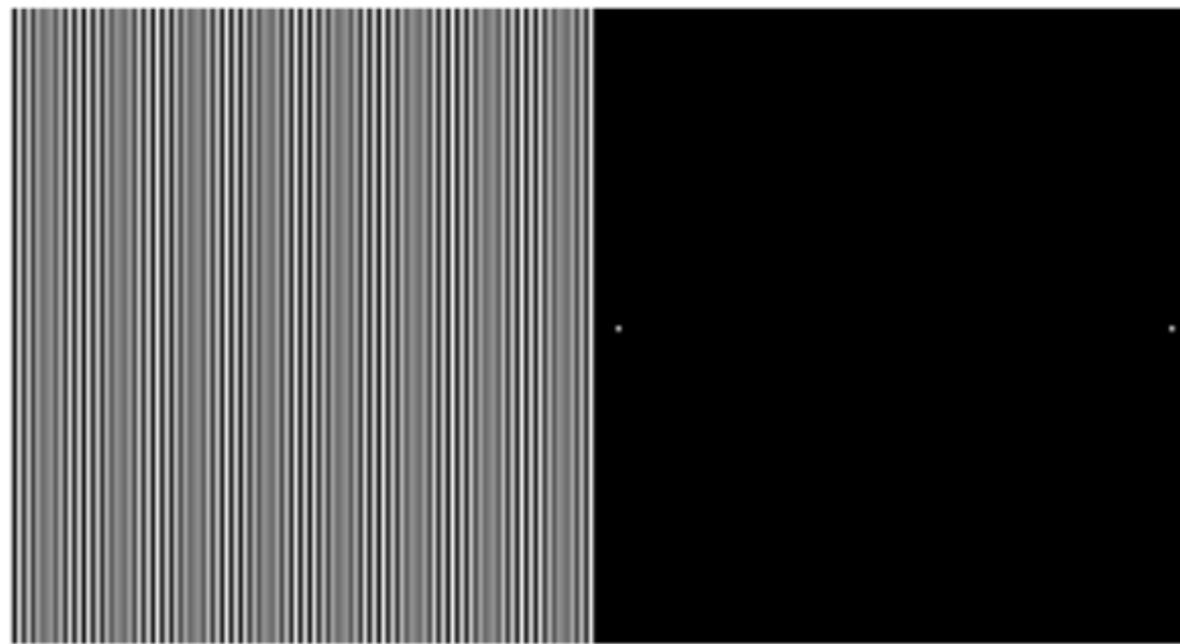
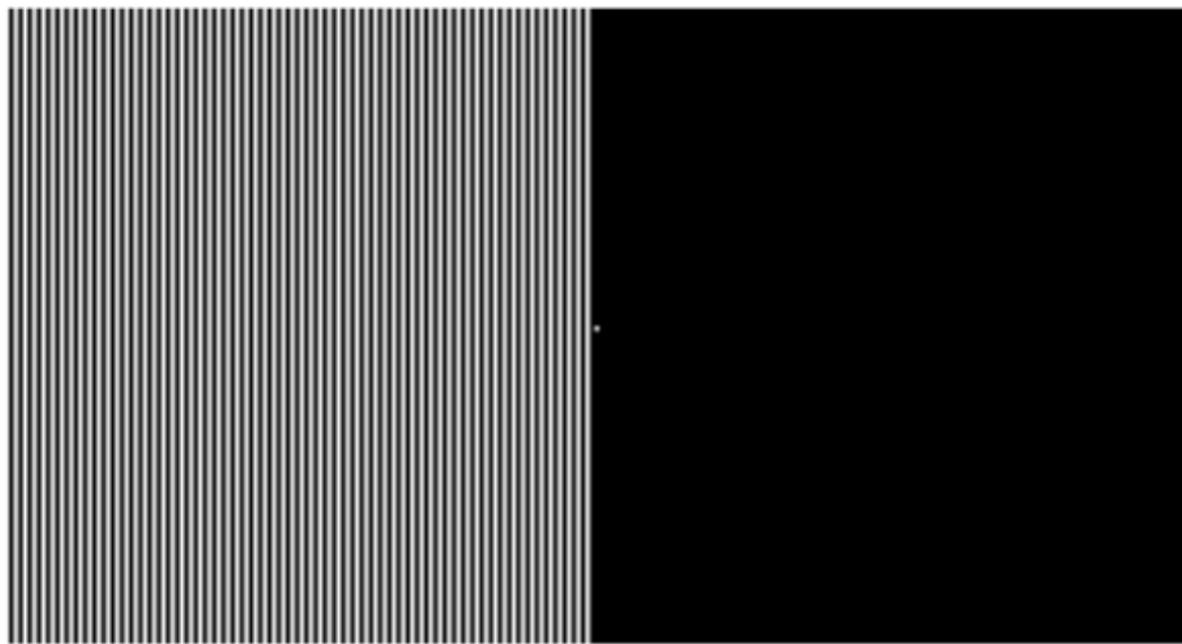




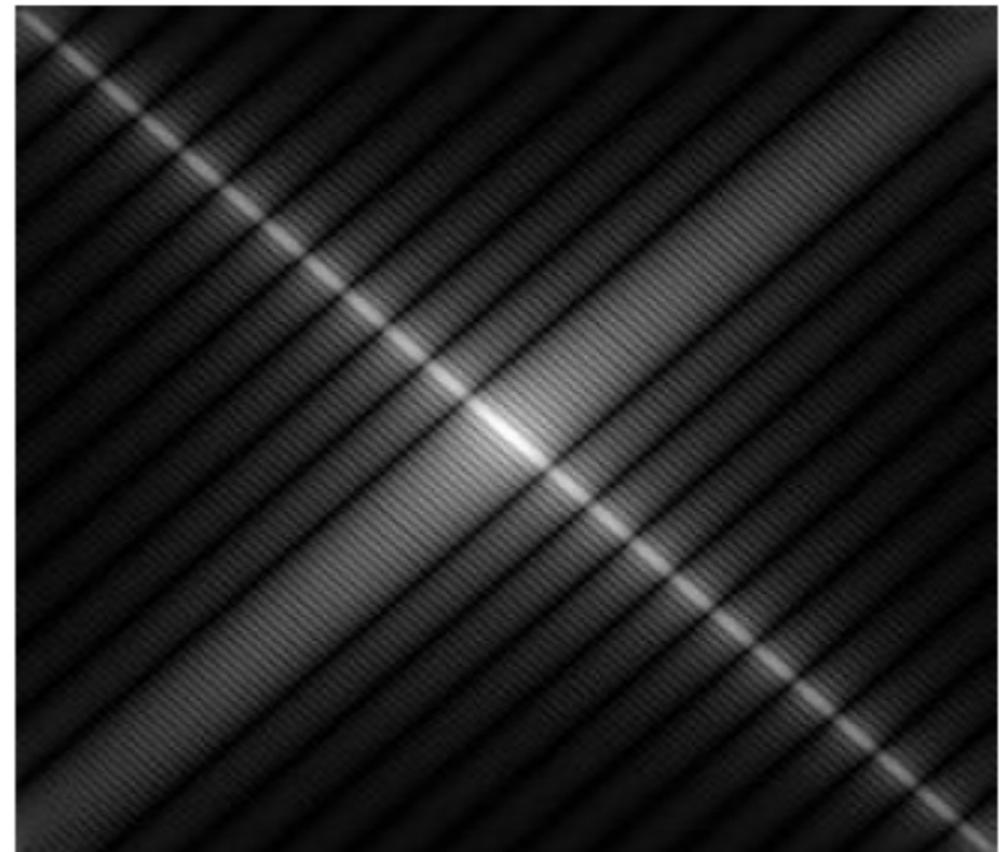
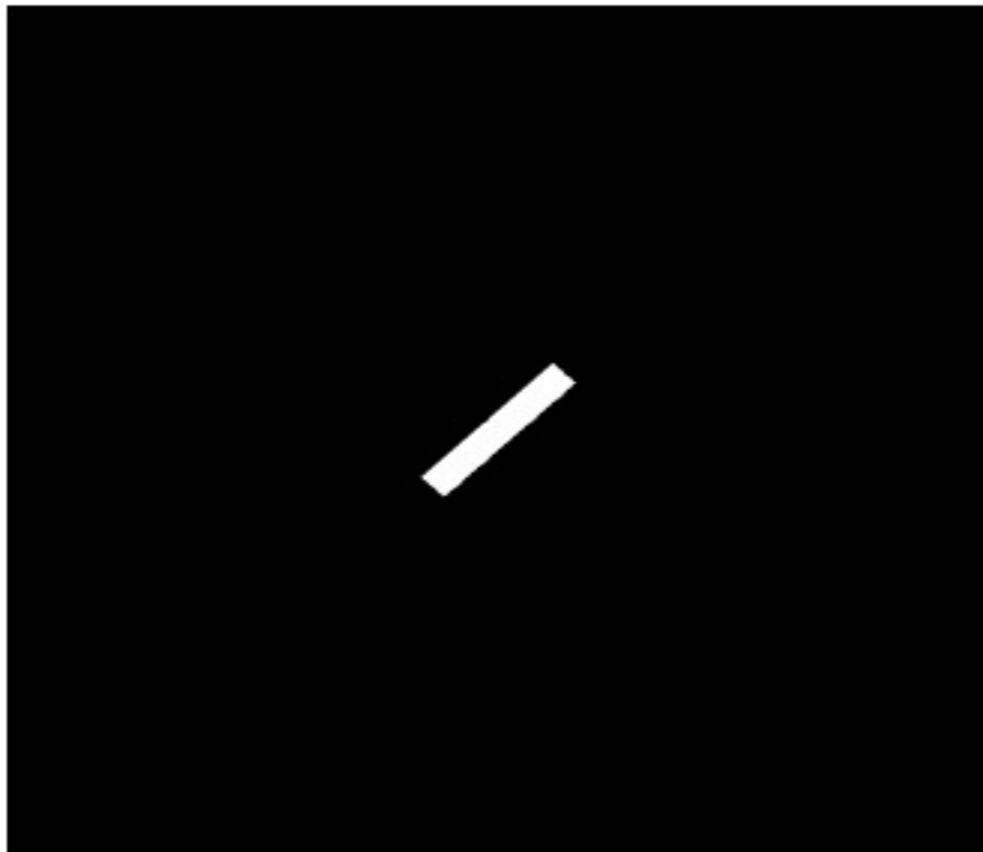
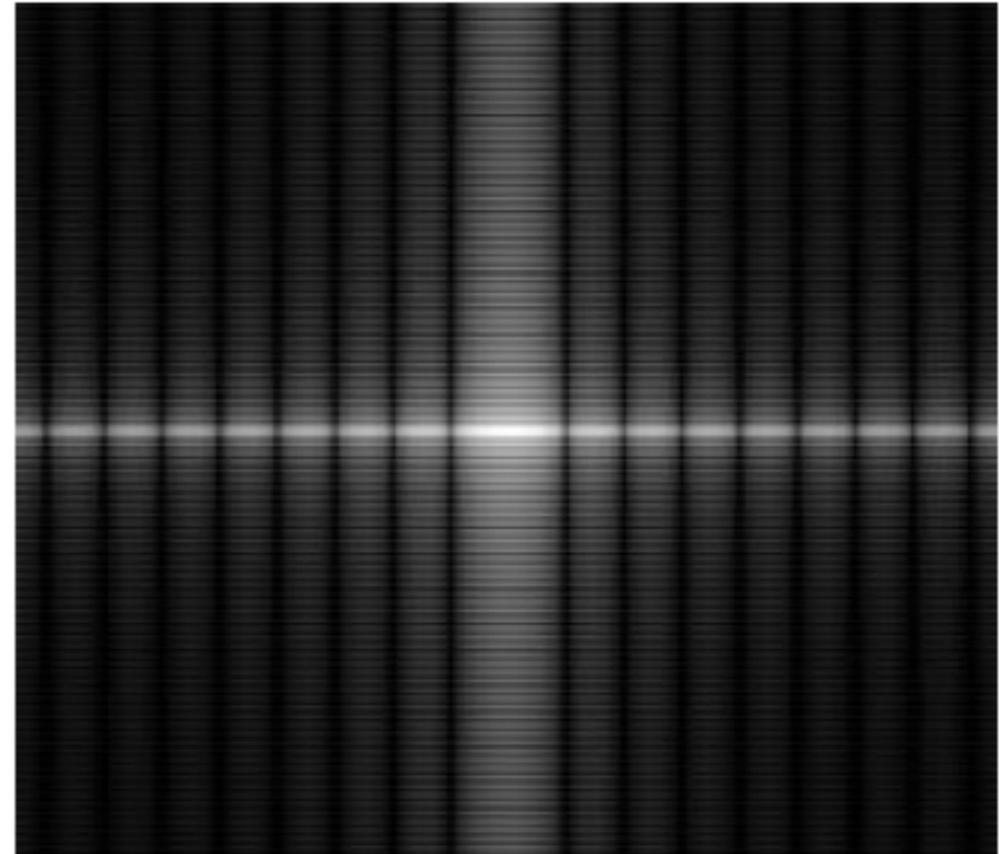
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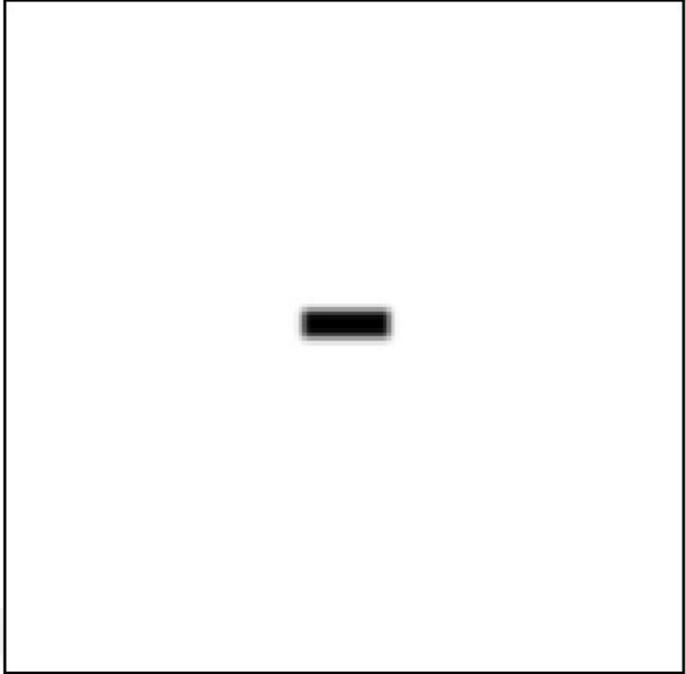
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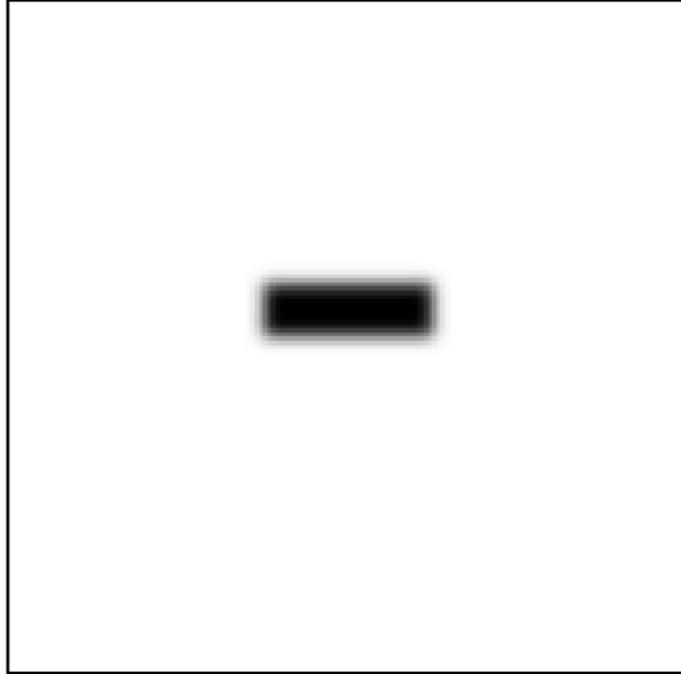
Rotación y traslación



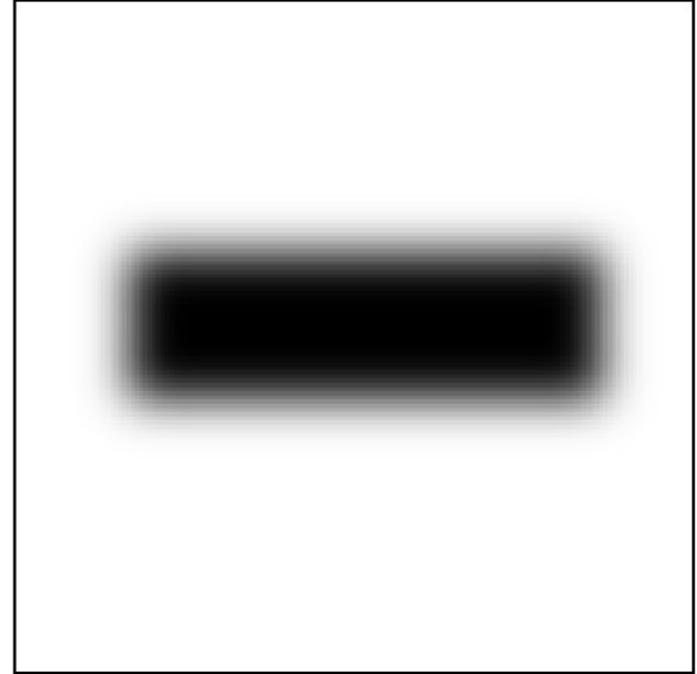
Ancho de banda



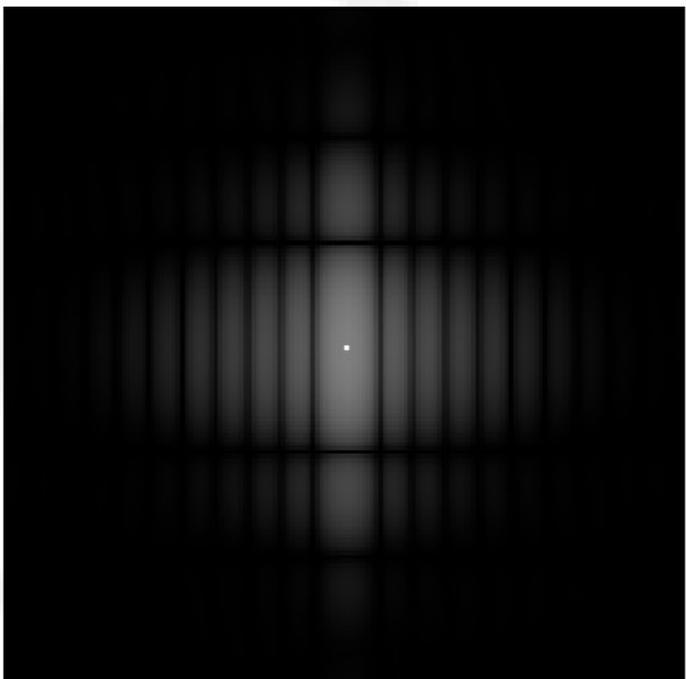
(a)



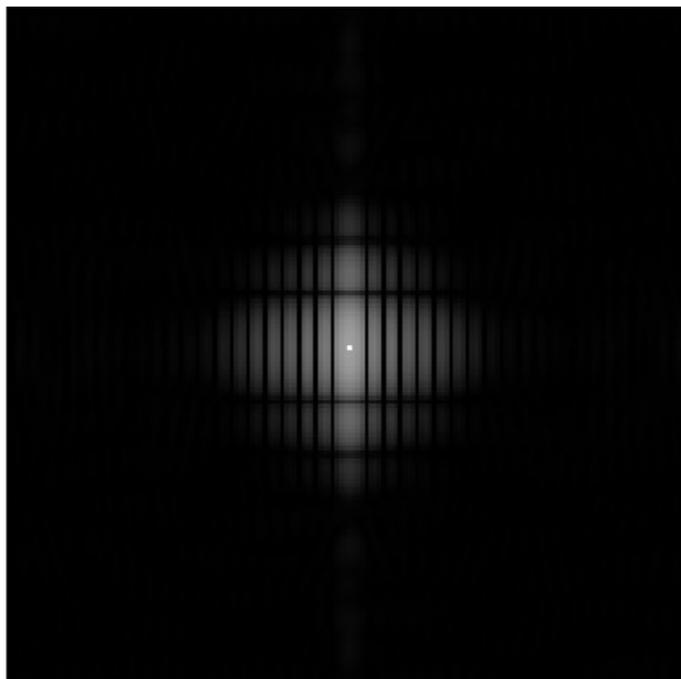
(b)



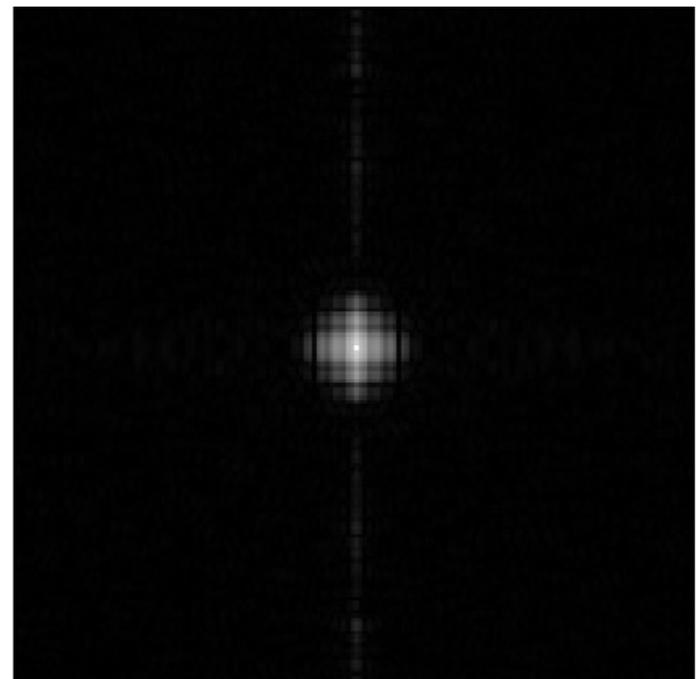
(c)



(d)

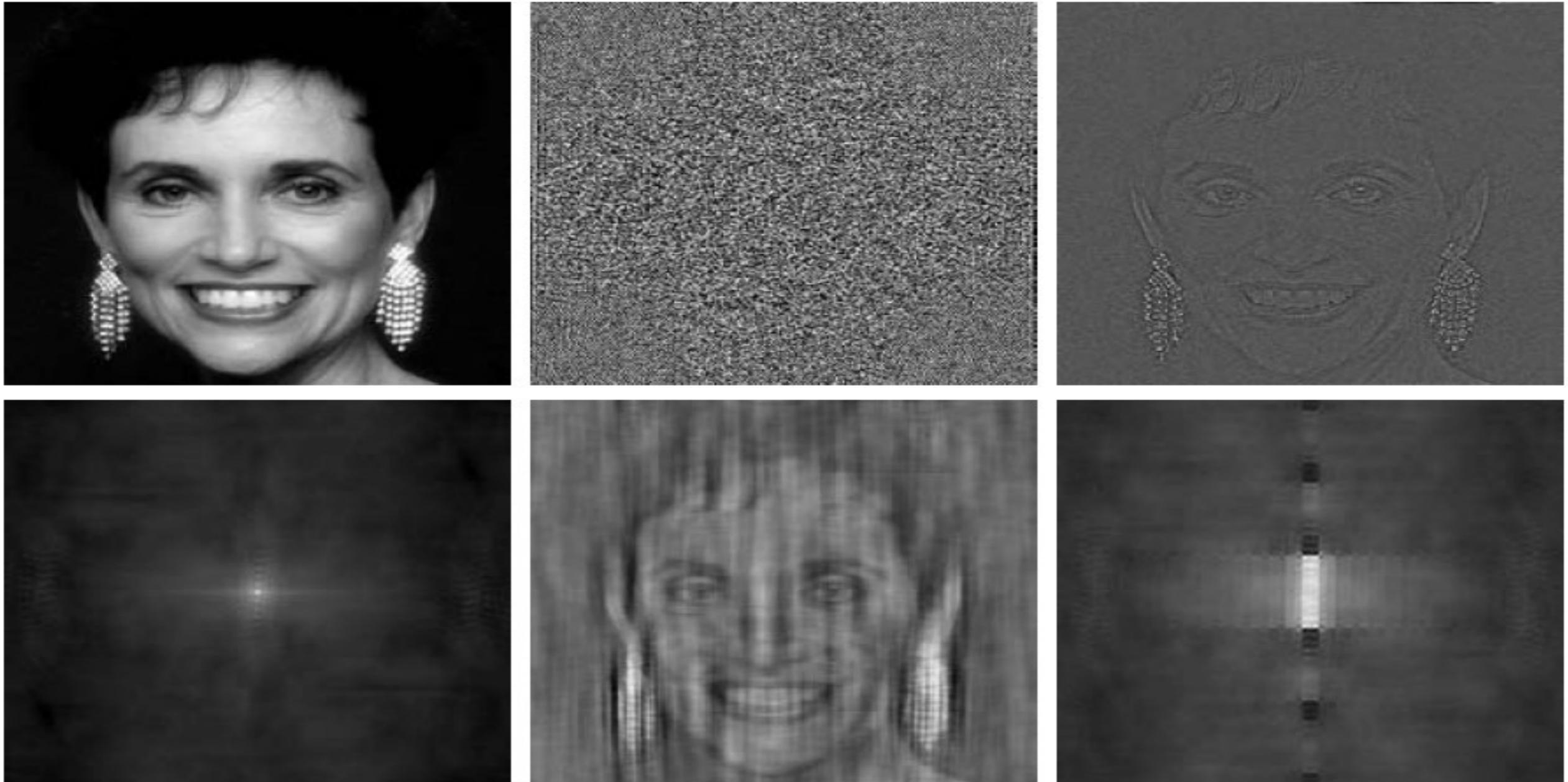


(e)



(f)

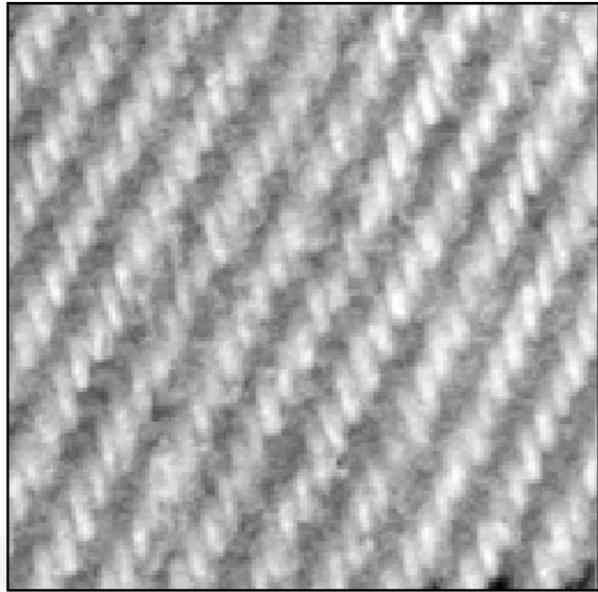
Módulo y fase



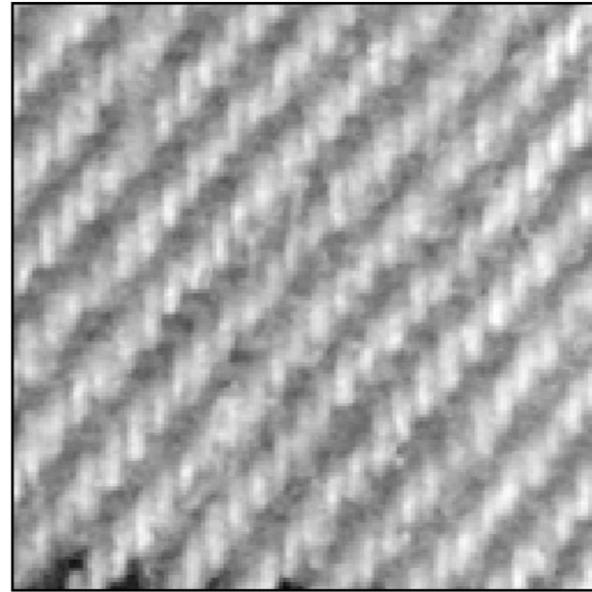
a	b	c
d	e	f

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

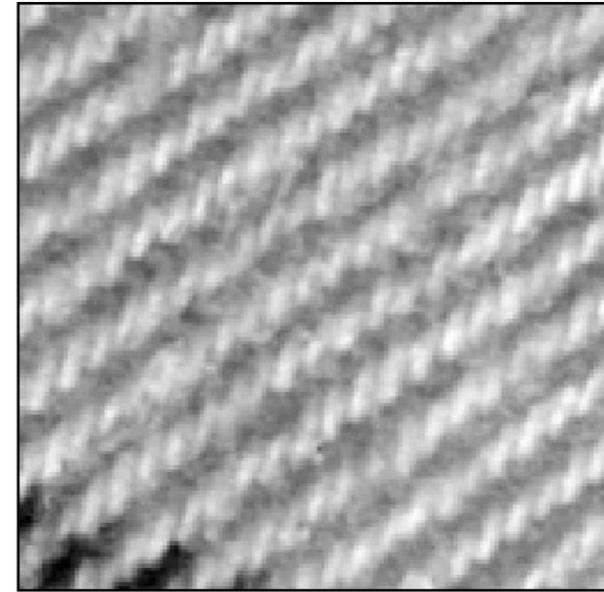
Ejemplos



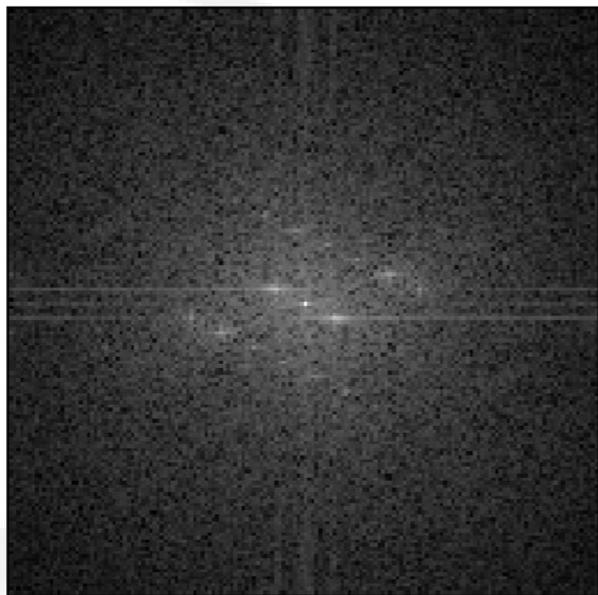
(a)



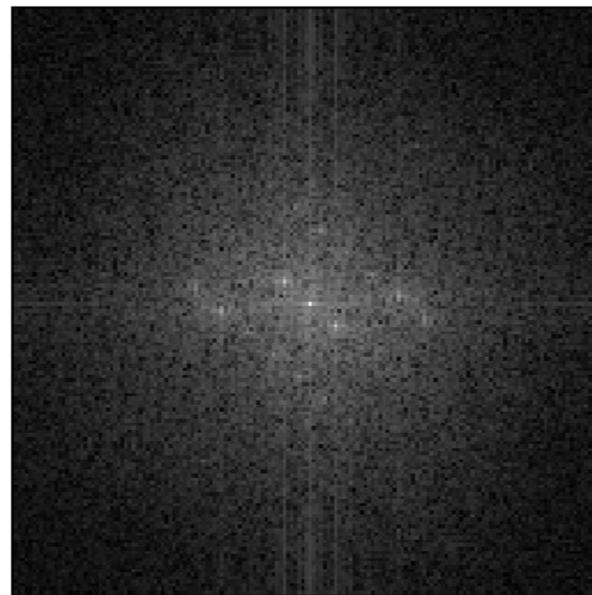
(b)



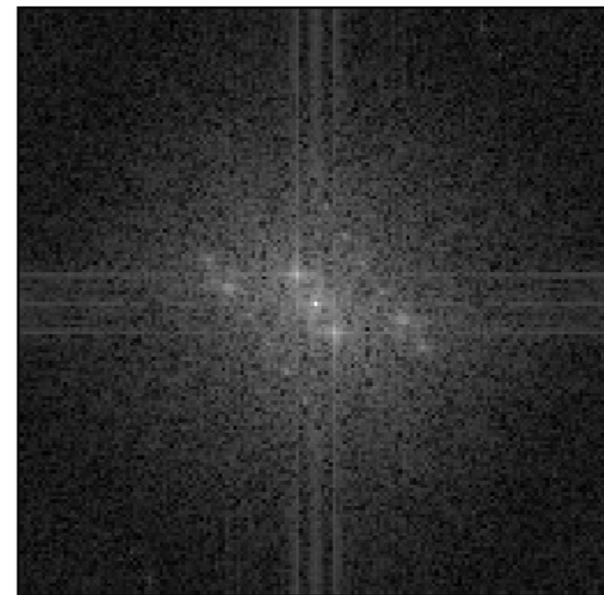
(c)



(d)

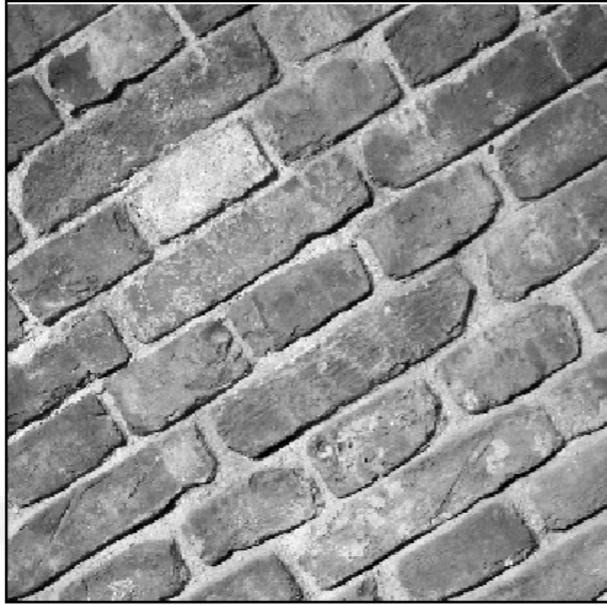


(e)



(f)

Ejemplos



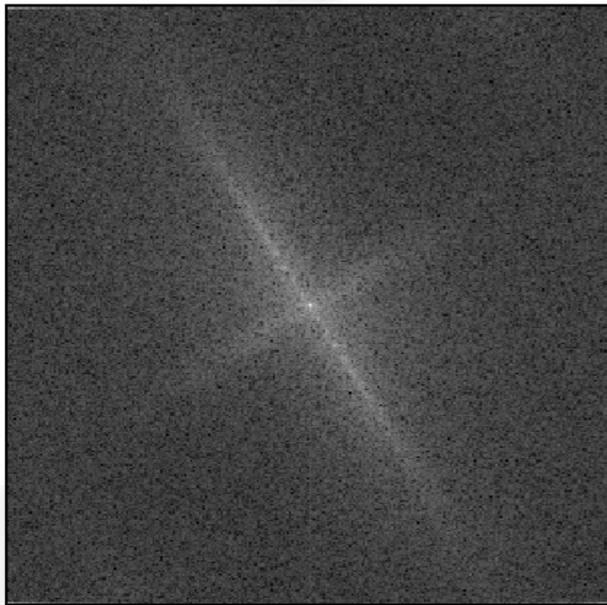
(a)



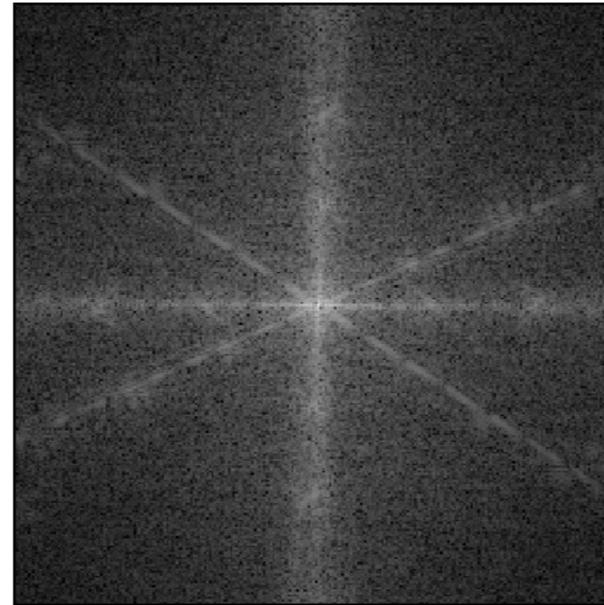
(b)



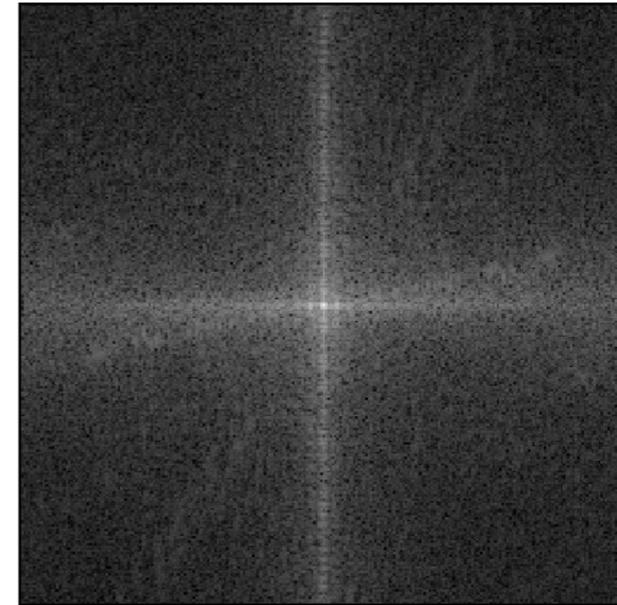
(c)



(d)



(e)



(f)

Ejemplos



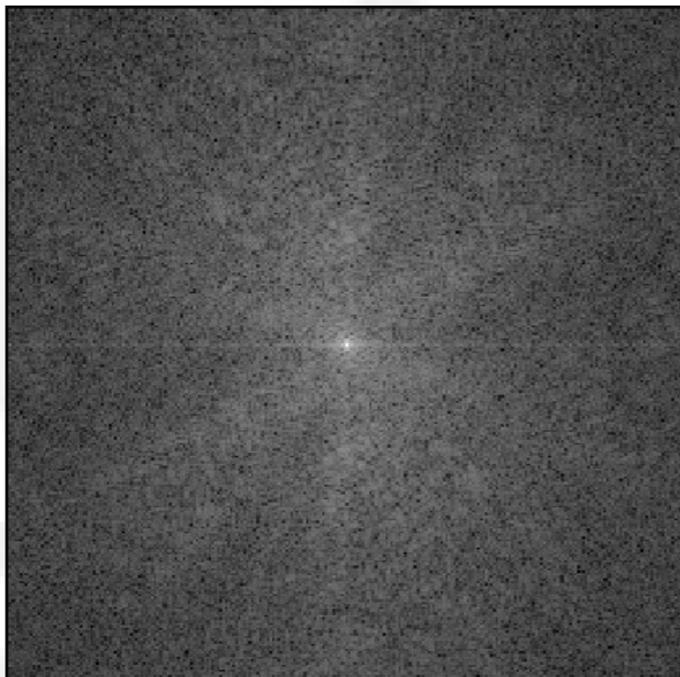
(a)



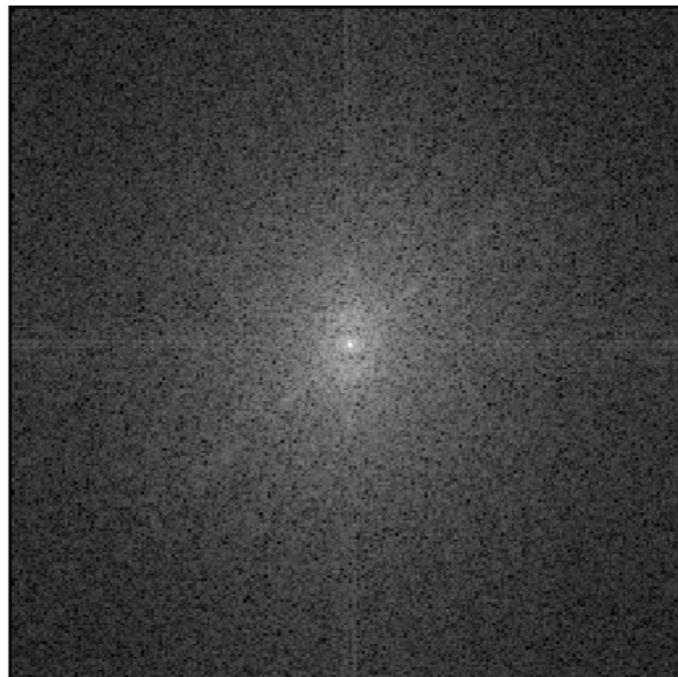
(b)



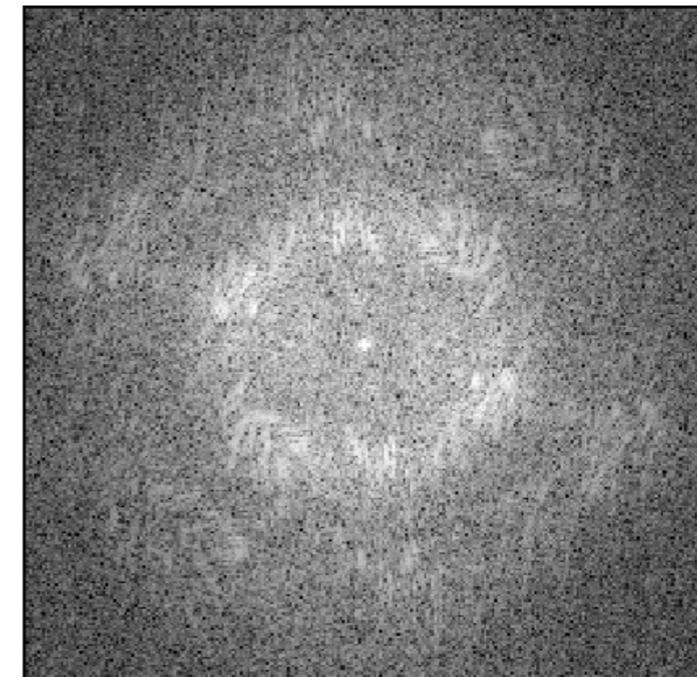
(c)



(d)



(e)



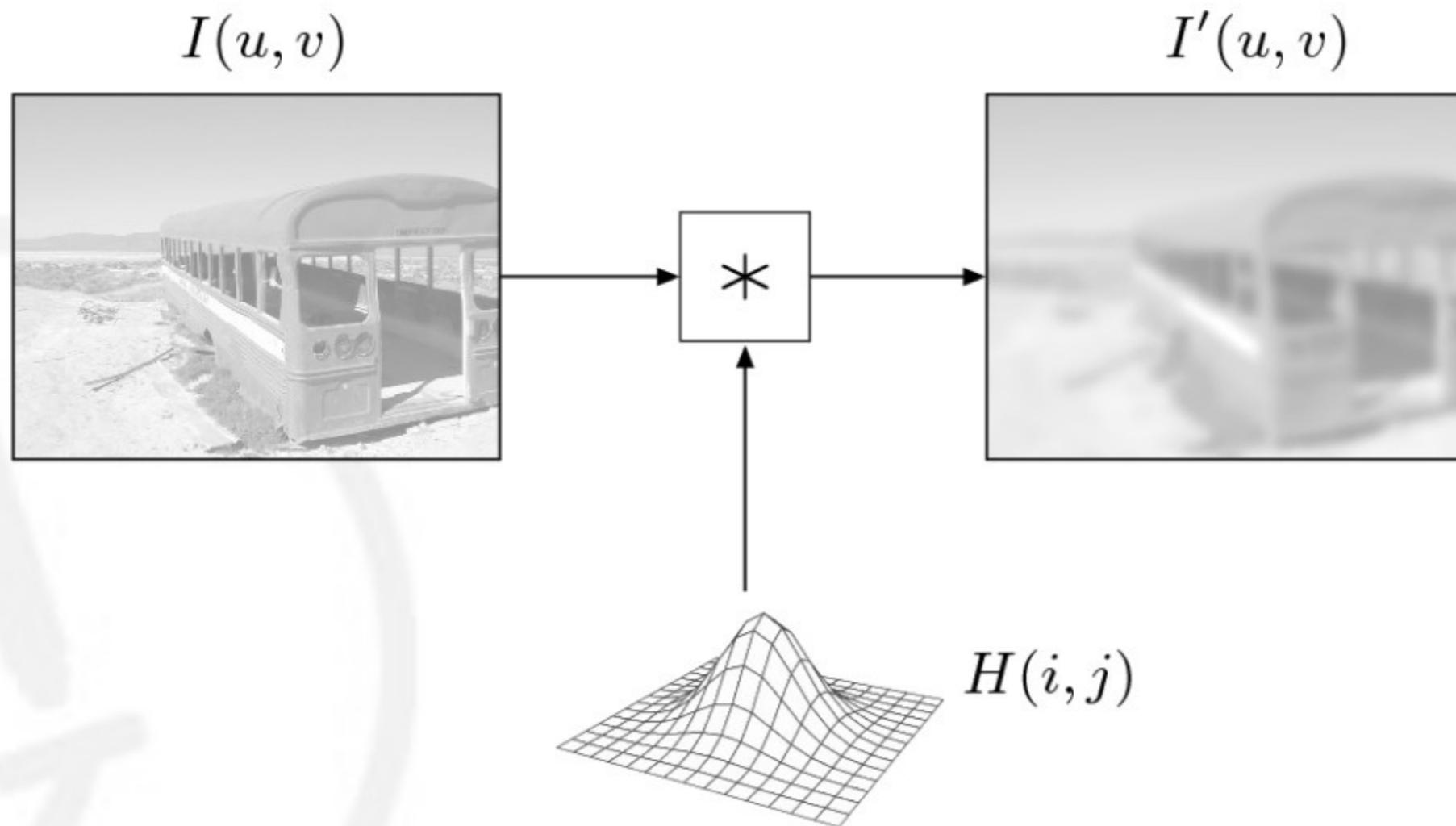
(f)

Filtrado en el espacio y en frecuencia

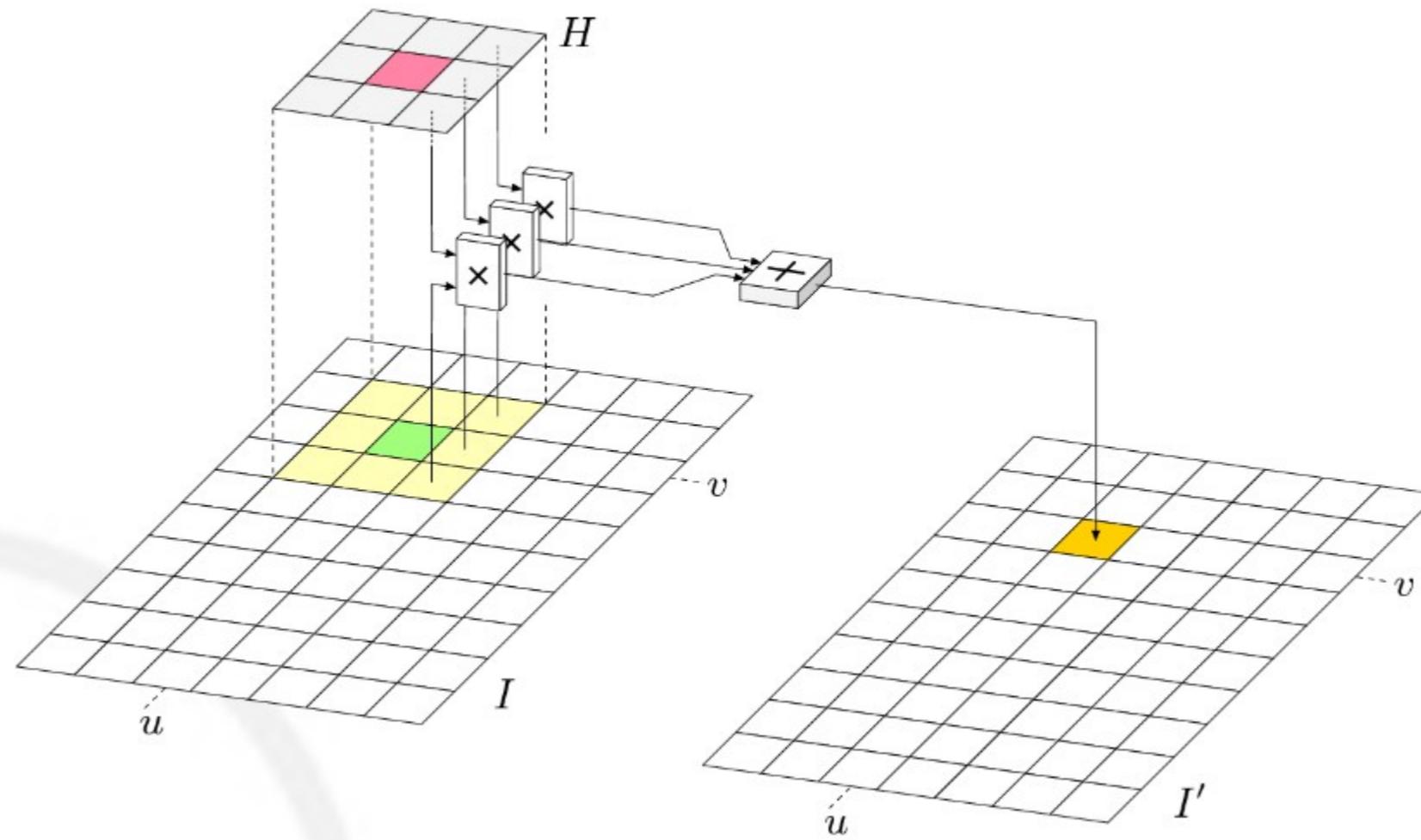


Filtrado lineal

- El filtrado lineal es la convolución entre la imagen y el núcleo H



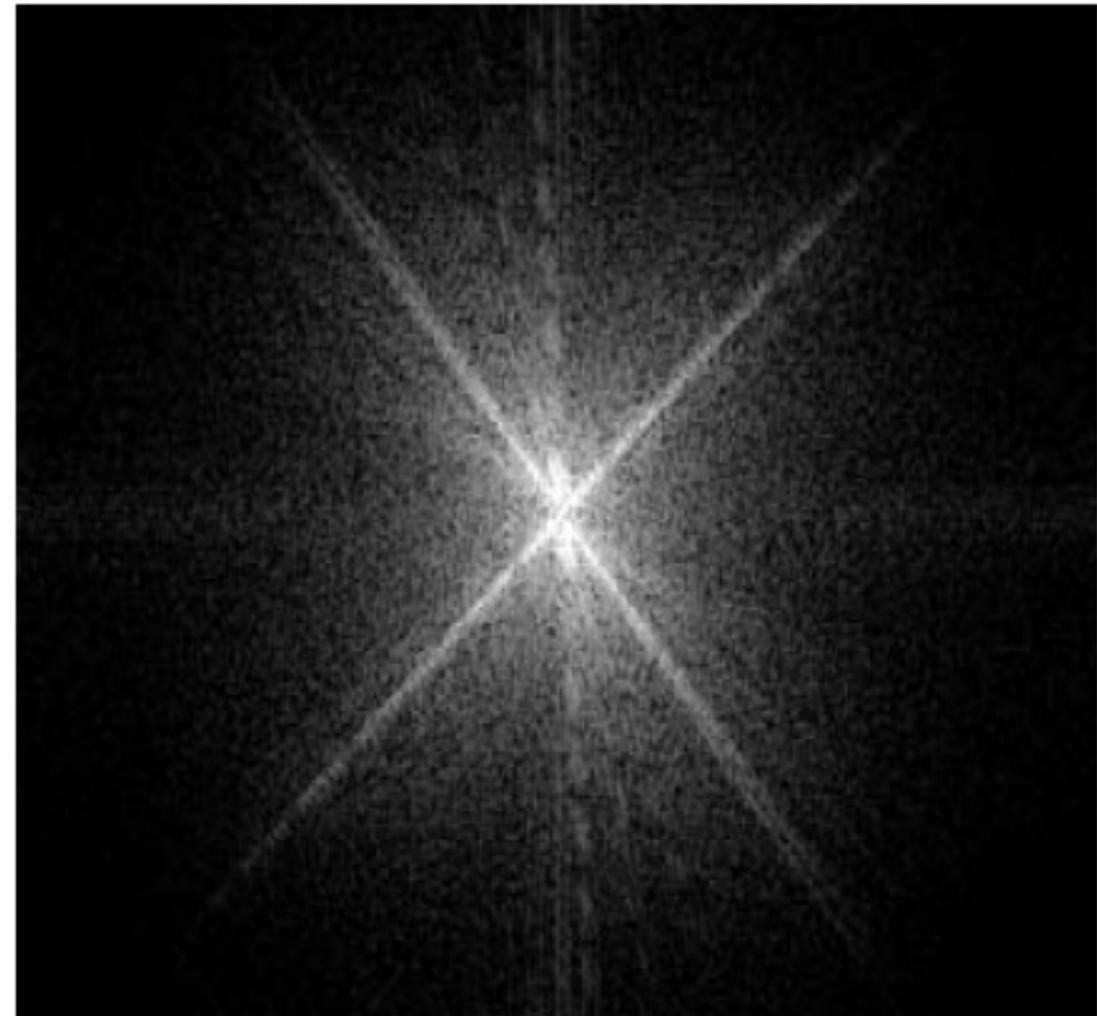
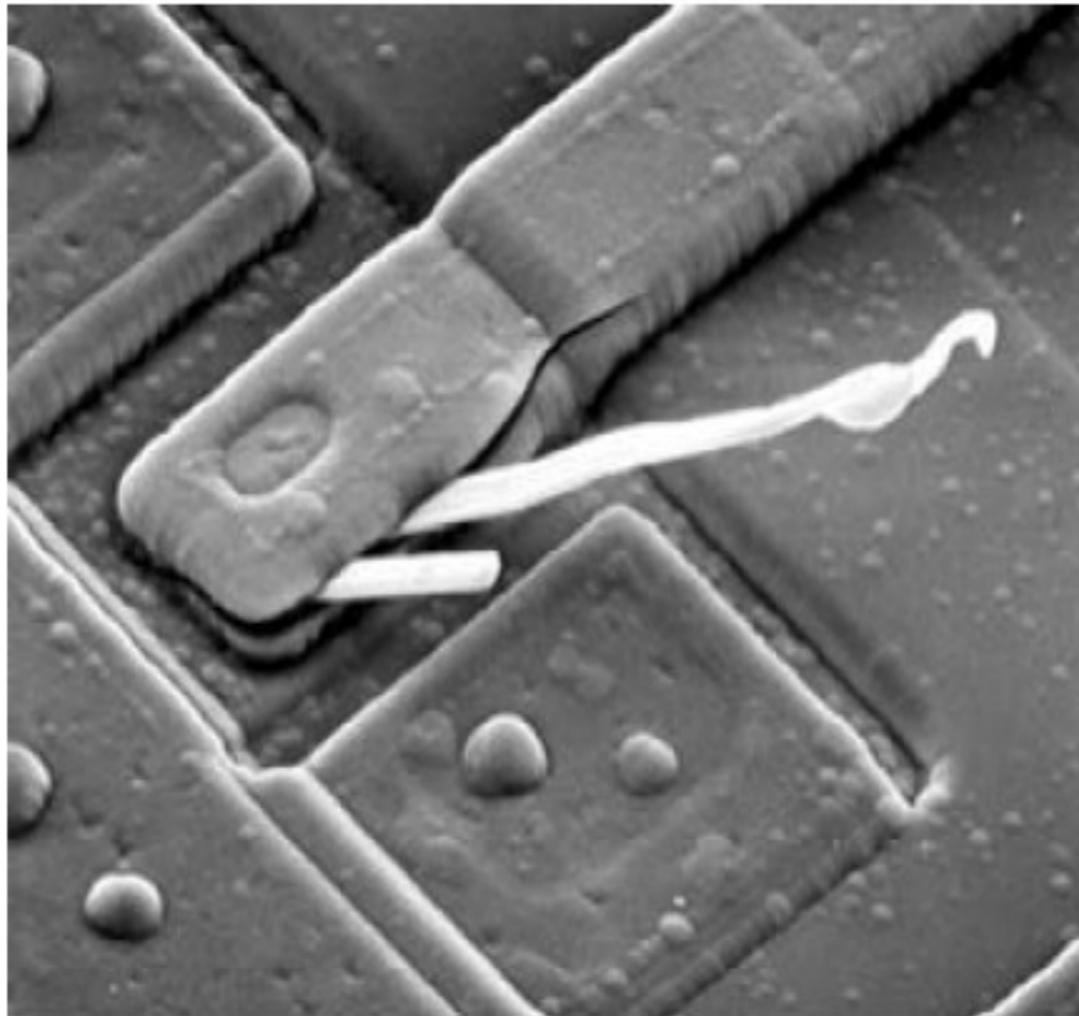
Filtrado lineal



Teorema de convolución

- Propiedad importante de la transformada de Fourier
 - A una operación de convolución en el espacio le corresponde una multiplicación puntual en frecuencia

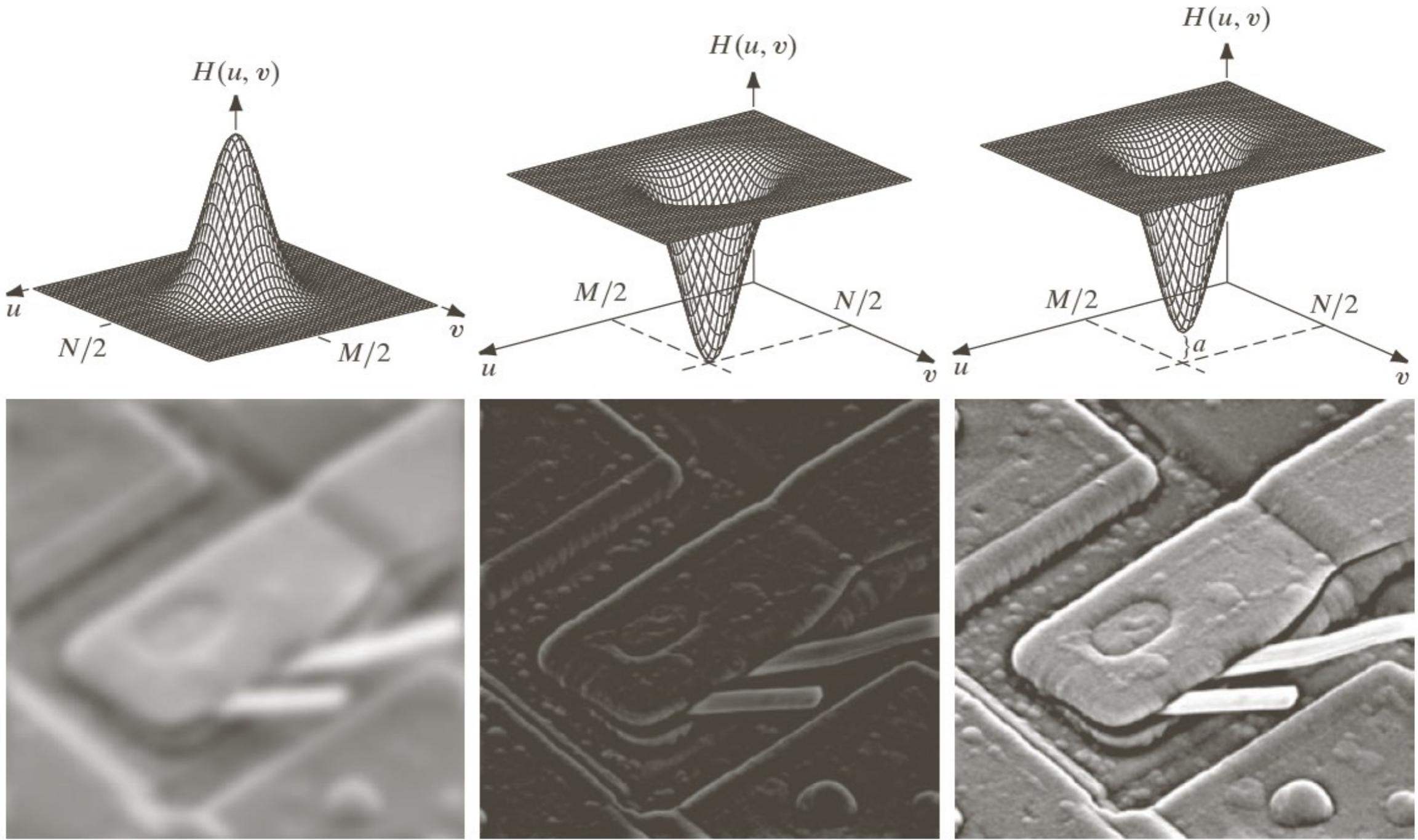
$$\begin{array}{ccccc} \text{Image space: } & g(u, v) & * & h(u, v) & = & g'(u, v) \\ & \downarrow & & \downarrow & & \uparrow \\ & \text{DFT} & & \text{DFT} & & \text{DFT}^{-1} \\ & \downarrow & & \downarrow & & \uparrow \\ \text{Frequency space: } & G(m, n) & \cdot & H(m, n) & \longrightarrow & G'(m, n) \end{array}$$



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

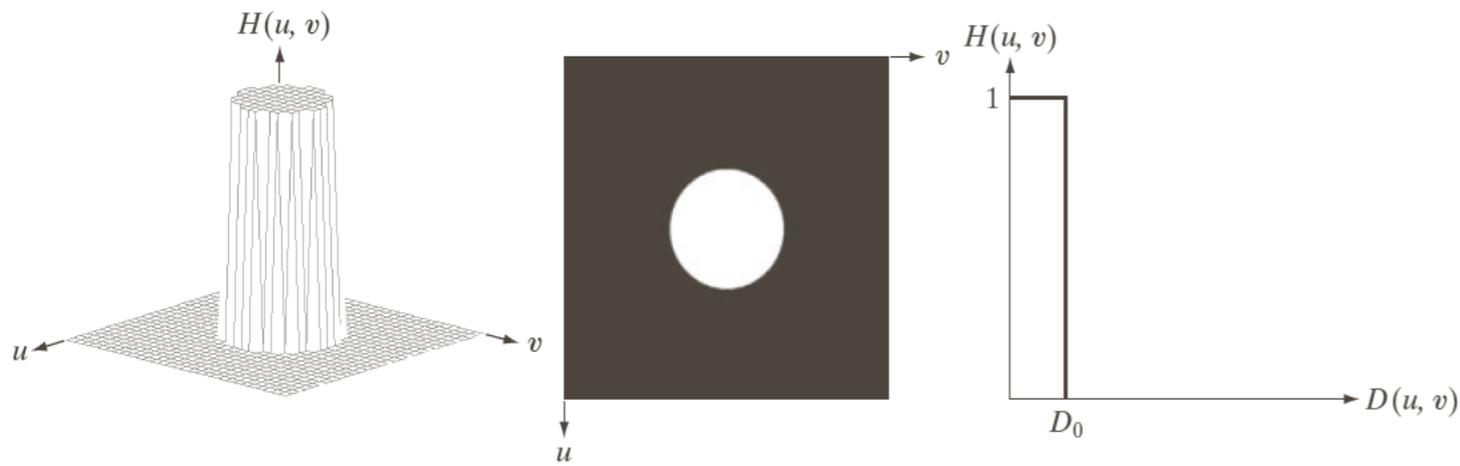
Filtrado en frecuencia



a	b	c
d	e	f

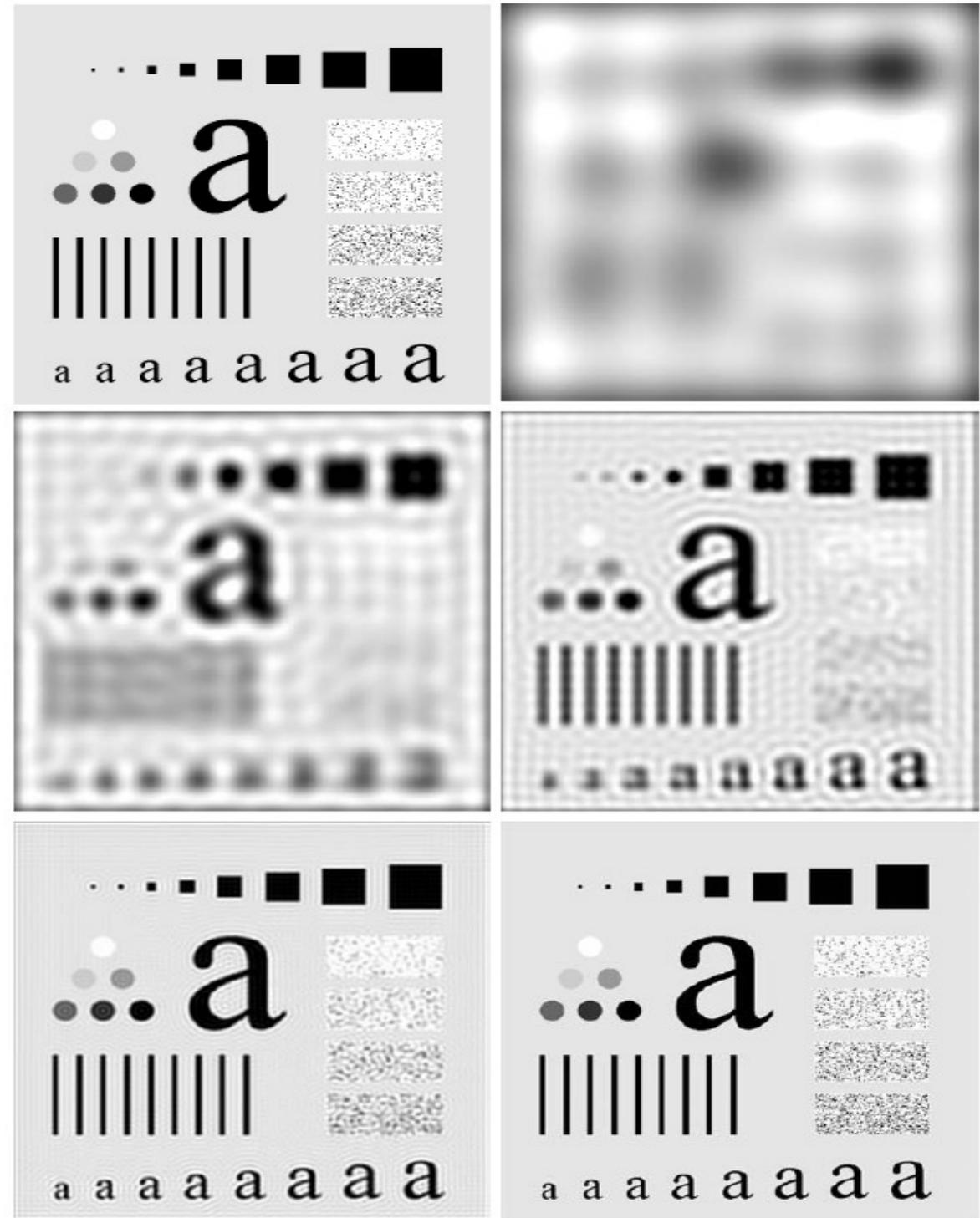
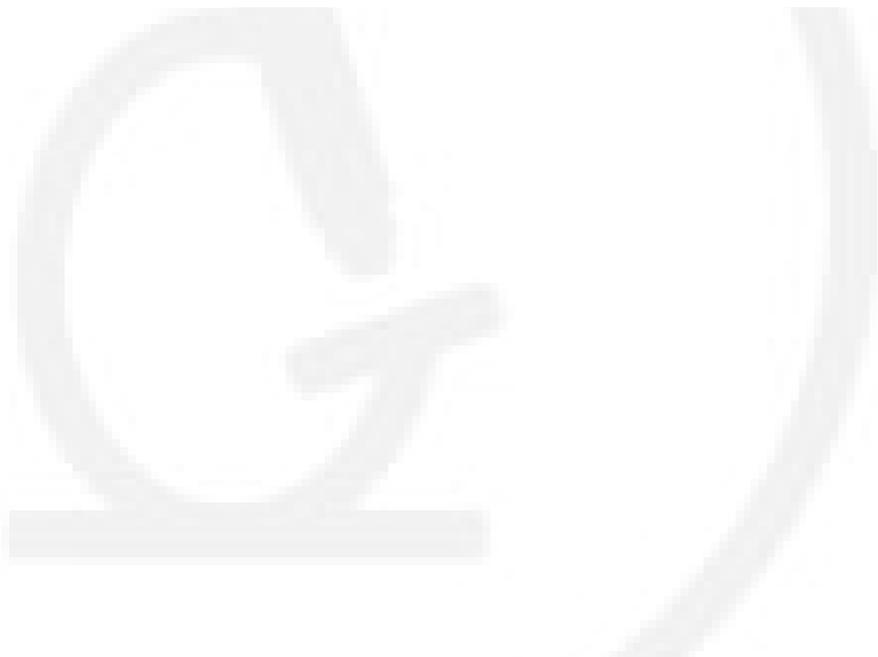
FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Filtrado pasabajos "ideal"



a b c

FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.



a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Filtro pasabajos “suave”

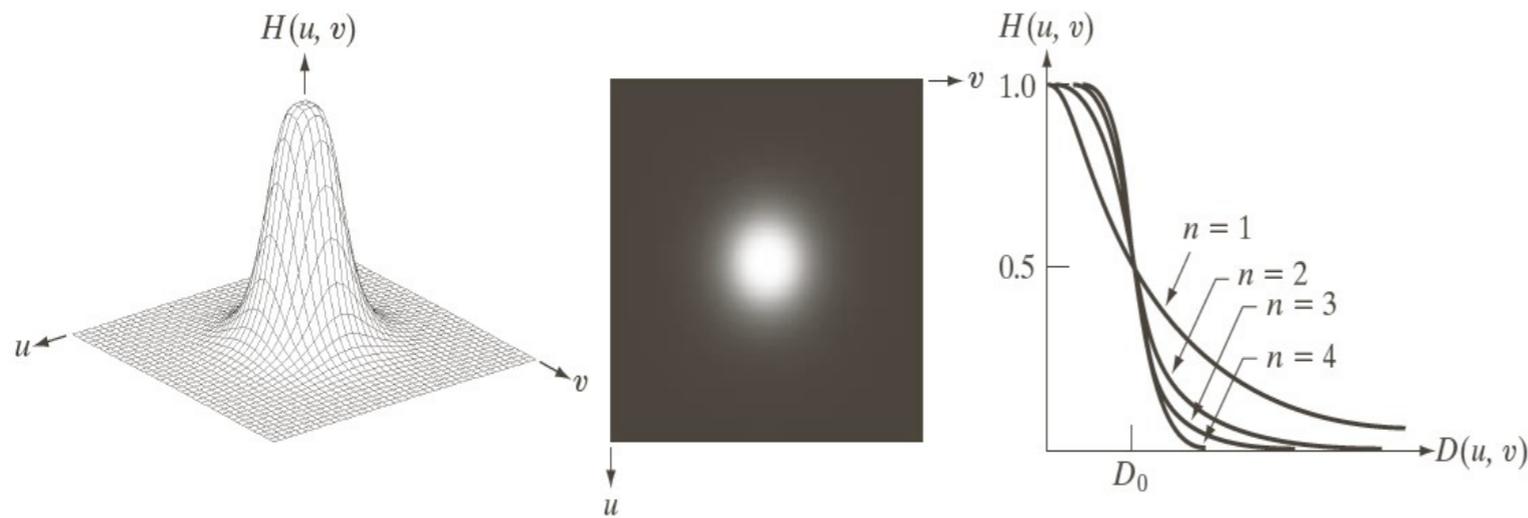


FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

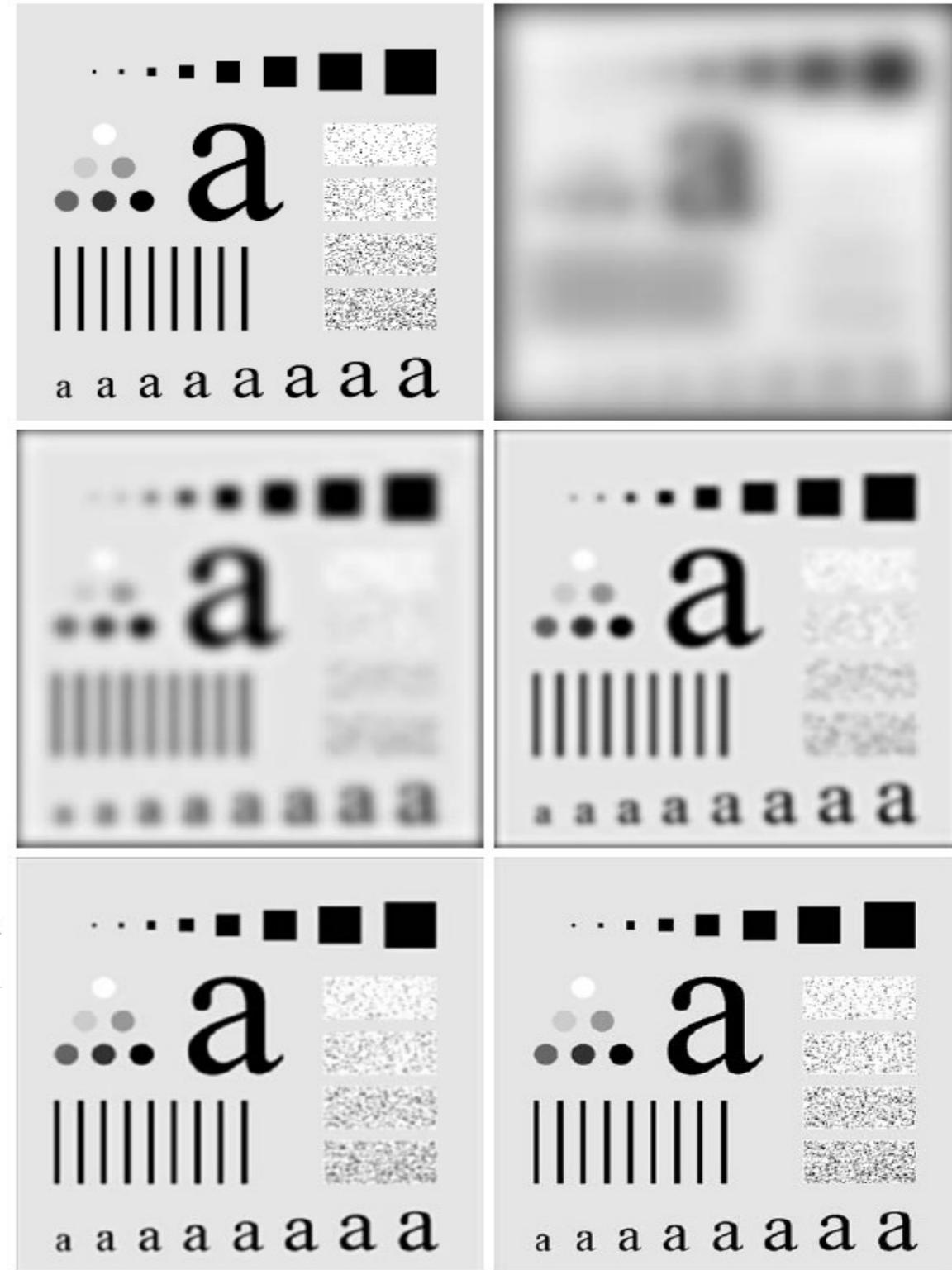
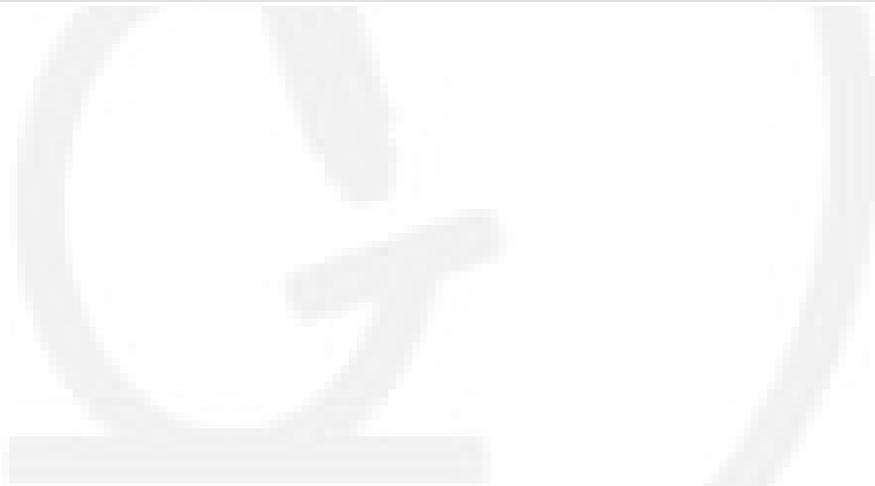
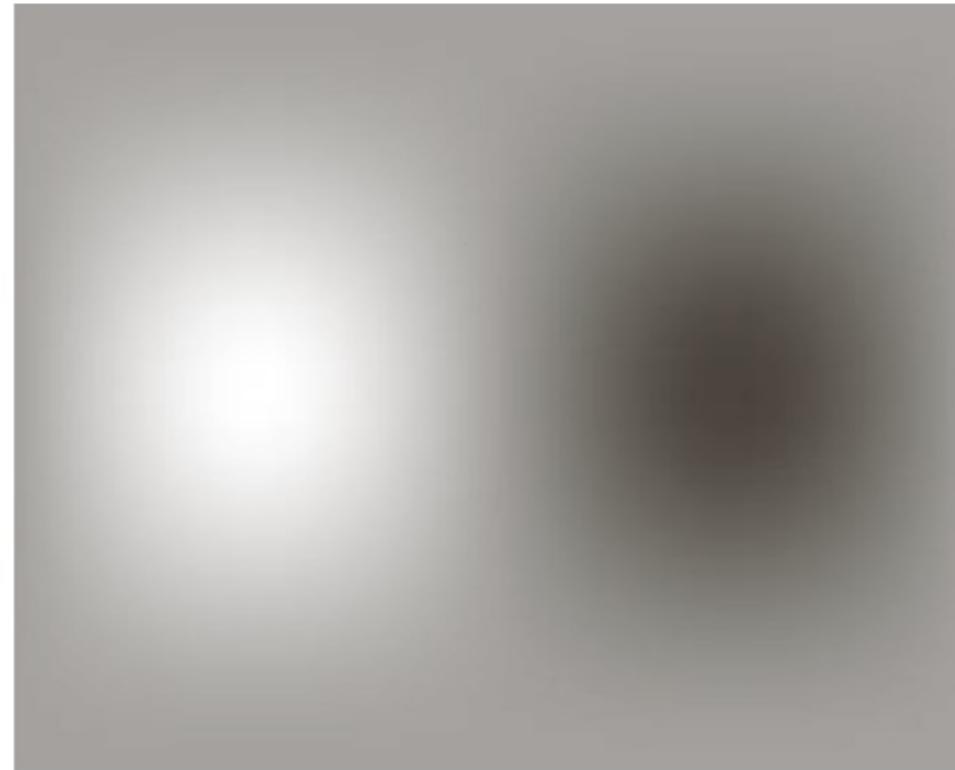
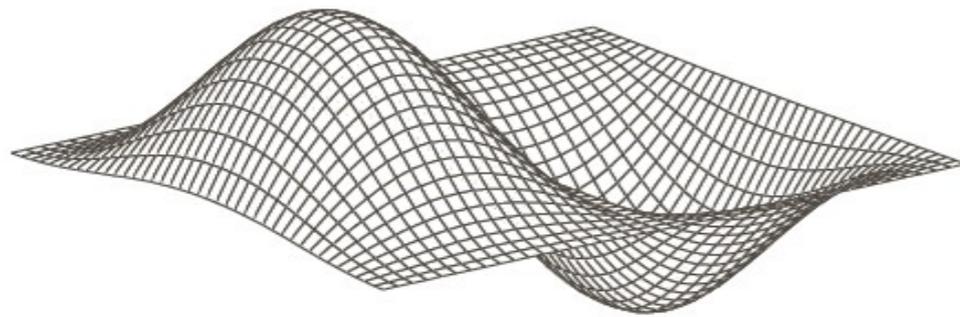


FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.



Filtrado espacial – filtrado frecuencial

-1	0	1
-2	0	2
-1	0	1



a	b
c	d

FIGURE 4.39
(a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

Transformada de Fourier Fiji

