Clase 38:

Cambio de variable

CDIVV - 2023 - 2sem

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Teorema de cambio de variable

Sea $f: \underset{\mathbb{R}^2}{\overset{\cap}{D}} \longrightarrow \mathbb{R}$ continua y

$g: U \longrightarrow V$ un cambio de variable

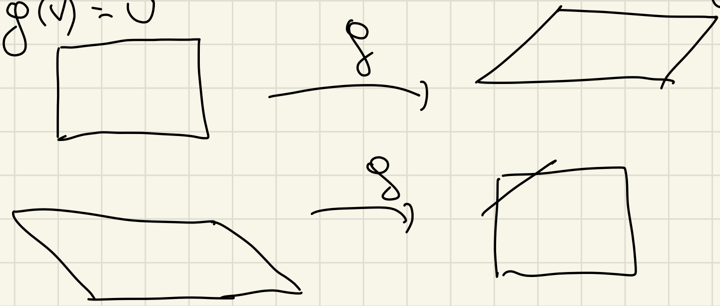
con $D \subseteq V$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{g^{-1}(D)} f(g(u,v)) \cdot \left| \det J_g(u,v) \right| du\,dv}$$

Obs $f(x,y) = 1$ $\qquad g(v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot v$

$g: U \longrightarrow V$ c.V lineal

$g^{-1}(V) = U$ $\qquad V = g(U)$



$$A(D) = A(g^{-1}(D)) \cdot \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

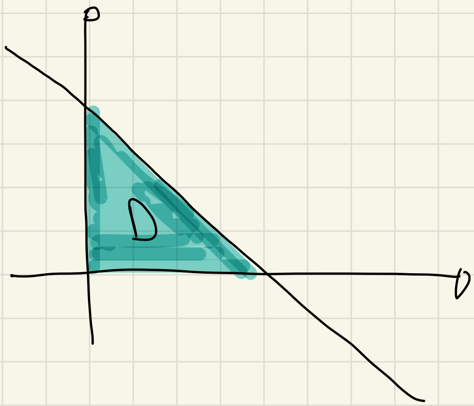
$$A(g(R)) = A(R) \cdot \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Ejemplo:

$$\iint_D e^{\frac{x-y}{x+y}} \, dx\,dy$$

u, v

$$D = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$$



Vamos a hacer un cambio de variable

$u = x - y$

$v = x + y$

$h(x,y) = (x-y, x+y) = (u,v)$

$\shortparallel$

g^{-1}

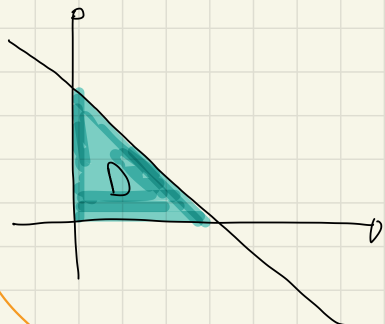
es un cambio de variable lineal

$$(x,y) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{g^{-1}(D)} f(g(u,v)) \cdot |\det J_g(u,v)|\,du\,dv}$$

$$f(x,y) = e^{\frac{x-y}{x+y}}$$

$$D = \{(x,y) \in \mathbb{R}^2 ;\ x \geq 0,\ y \geq 0,\ x+y \leq 1\}$$



$$\begin{cases} u = x+y \\ v = x-y \end{cases} \qquad x = \frac{u+v}{2} \qquad y = \frac{u-v}{2}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

$$g(u,v) = \Big(\underbrace{\frac{u+v}{2}}_{x},\ \underbrace{\frac{u-v}{2}}_{y}\Big) = \underbrace{\begin{pmatrix} \frac{1}{2} & 1/2 \\ 1/2 & -1/2 \end{pmatrix}}_{J_g(u,v)}$$

$$f(x,y) = e^{\frac{x-y}{x+y}}$$

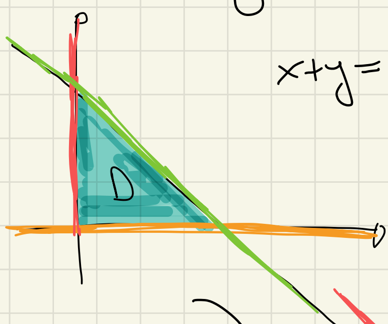
$$f(g(u,v)) = f\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$$= e^{\frac{\frac{u+v}{2} - \frac{u-v}{2}}{\frac{u+v}{2} + \frac{u-v}{2}}}$$

$$= e^{\frac{v}{u}}$$

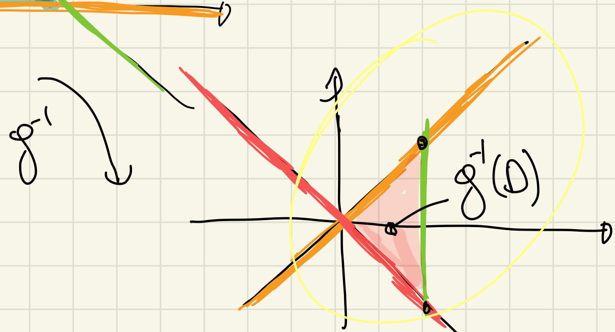
$g^{-1}(D)$

Le aplicamos $g^{-1}(x,y) = (x-y, x+y)$ a D



$x+y=1 \quad y=1-x$

$g^{-1}(x, 1-x) = (x-1+x, 1)$



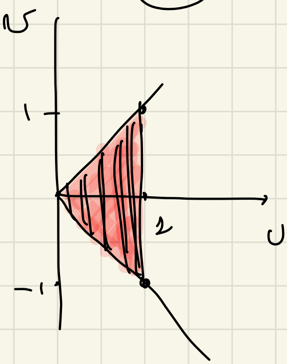
$$\left|\det J_\rho(u,v)\right| = \left|\det\begin{pmatrix}1/2 & 1/2\\ 1/2 & -1/2\end{pmatrix}\right| = \left|-\frac{1}{4}-\frac{1}{4}\right|$$

$$= \left|-\frac{1}{2}\right| = \frac{1}{2}$$

$$\iint_D e^{\frac{x-y}{x+y}}\,dx\,dy \underset{\text{C.V}}{=} \iint_{\rho^{-1}(D)} e^{\frac{v}{u}}\,\frac{1}{2}\,du\,dv$$

$$=$$

$\rho^{-1}(D) =$



$$\underset{\text{TF}}{=} \int_0^1\left(\int_{-u}^{u}\frac{1}{2}e^{\frac{v}{u}}\,dv\right)du = \frac{1}{2}\int_0^1 u e^{\frac{v}{u}}\Big|_{-u}^{u}\,du$$

$$= \frac{1}{2}\int_0^1 u\left(e^{\frac{u}{u}} - e^{\frac{-u}{u}}\right)du$$

$$= \frac{1}{2} \int_0^1 u\left(e - \frac{1}{e}\right) du$$

$$= \left(e - \frac{1}{e}\right)\left(\frac{1}{2}\right) \frac{u^2}{2}\Big|_0^1 = \left(e - \frac{1}{e}\right)\frac{1}{4}$$

Coordenadas polares.

$x = \rho \cos\theta$

$y = \rho \operatorname{sen}\theta$

$$g(\rho, \theta) = (\rho \cos\theta, \rho \operatorname{sen}\theta)$$

$$J_g(\rho, \theta) = \begin{pmatrix} \cos\theta & -\rho \operatorname{sen}\theta \\ \operatorname{sen}\theta & \rho \cos\theta \end{pmatrix}$$

$$|J_g(\rho, \theta)| = \left|\rho \cos^2\theta - \left(-\rho \operatorname{sen}^2\theta\right)\right|$$

$$= \left|\rho \cos^2\theta + \rho \operatorname{sen}^2\theta\right| = |\rho|$$

$$= \rho$$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{g^{-1}(D)} f\circ g(\rho,\theta)\cdot\rho\,d\rho\,d\theta}$$

Ejemplo:

$$\iint_D e^{x^2+y^2}\,dx\,dy \qquad D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 1\}$$

$$x=\rho\cos\theta$$

$$y=\rho\,\mathrm{sen}\,\theta$$

$$(f\circ g)(\rho,\theta) = e^{\rho^2\cos^2\theta+\rho^2\mathrm{sen}^2\theta} = e^{\rho^2}$$

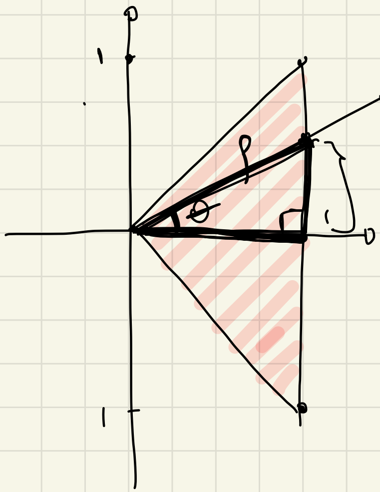
$$g^{-1}(D)=\{(\rho,\theta):\ \rho\leq 1,\ \theta\in[0,2\pi)\}.$$

$$\iint_{\{(\rho,\theta):\ 0\leq\rho\leq 1,\ \theta\in[0,2\pi)\}} e^{\rho^2}\,\rho\,d\rho\,d\theta$$

$$\overset{TF}{=} \int_0^{2\pi} \left(\int_0^1 e^{\rho^2} \rho \, d\rho \right) d\theta = \ldots$$

Ejemplo: Hallar $\displaystyle\iint_D \frac{x^2}{x^2+y^2}\,dx\,dy$

$D =$



$$\cos\theta = \frac{1}{\rho}$$

$$\rho = \frac{1}{\cos\theta}$$

c.v polares

$x = \rho\cos\theta$

$y = \rho\,\mathrm{sen}\,\theta$

$$\iint_D \frac{x^2}{x^2+y^2}\,dx\,dy = \iint_{g^{-1}(D)} \frac{\cancel{\rho^2}\cos^2\theta}{\cancel{\rho^2}\cos^2\theta + \cancel{\rho^2}\,\mathrm{sen}^2\theta}\,\rho \; d\theta\, d\rho$$

$$= \iint_{g^{-1}(D)} \cos^2\theta \, \rho \; d\theta \, d\rho$$

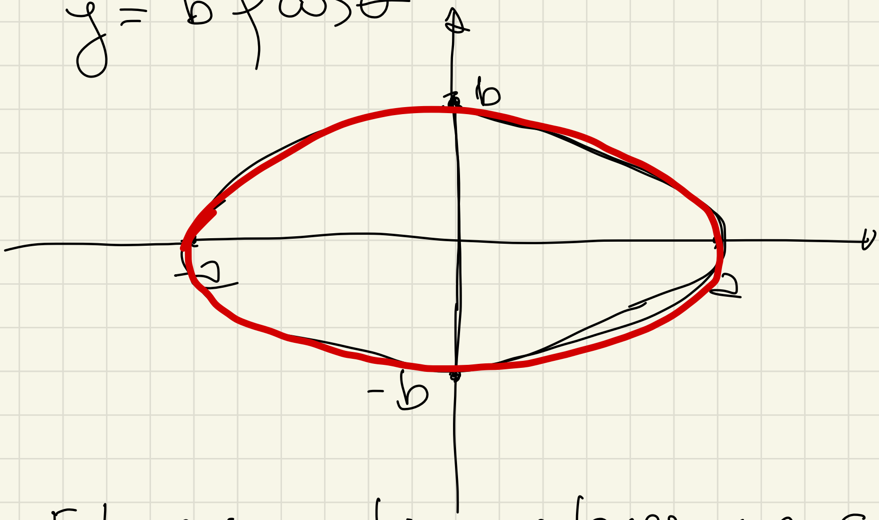
$$g^{-1}(D) = \left\{ (\rho, \theta) : (\rho\cos\theta, \rho \operatorname{sen}\theta) \in D \right\}.$$
$$= \left\{ (\rho, \theta) : -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \quad 0 \le \rho \le \frac{1}{\cos\theta} \right\}$$

$$\overset{TF}{=} \int_{-\pi/4}^{\pi/4} \left(\int_0^{1/\cos\theta} \cos^2\theta \, \rho \; d\rho \right) d\theta .$$

Coordenadas elípticas

$$x = a\rho\cos\theta$$

$$y = b\rho\cos\theta$$



Este c.v combina polares con c.v lineal.