Clase 35:

Desarrollo de Taylor

CDIVV - 2023 - 2sem

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Teorema de Taylor

- Sea $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ de clase C^{k+1} en un entorno de $a \in \mathbb{R}^m$.

Entonces

$$f(a+h) = f(a) + df_a(h) + \frac{1}{2} d^2 f_a(h) + \cdots + \frac{1}{k!} d^k f_a(h) + r_k(h)$$

en donde $\dfrac{r_k(h)}{\|h\|^k} \xrightarrow[h \to 0]{} 0$

$$d^n f_a(h_1, \dots, h_m) = \sum_{i_1, \dots, i_n = 1}^{m} \frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}}(a)\, h_{i_1} \cdots h_{i_n}$$

- Si $\boxed{m = 2}$ y $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ de clase C^{k+1} en un entoro de $a = (x_0, y_0) \in \mathbb{R}^2$

Entonces

$$x = x_0 + \Delta x = x_0 + h$$
$$y = y_0 + \Delta y = y_0 + k$$

$$f(x,y) = f(x_0+h, y_0+k) = f(x_0, y_0) + df_a(h,k)$$

$$+ \cdots + \frac{1}{k!} d^k f_a(h,k) + r_k(h,k)$$

con

$$d^k f_a(h,k) = \sum_{i=0}^{k} \binom{k}{i} \frac{\partial^k f(a)}{\partial^i x \, \partial^{k-i} y} h^i k^{k-i}$$

y

$$\frac{r_k(h,k)}{\|(h,k)\|^k} \xrightarrow[(h,k)\to 0]{} 0$$

$k=3$

$$d^3 f_a(h,k) = \sum_{i=0}^{3} \binom{3}{i} \frac{\partial^3 f}{\partial^i x \, \partial^{3-i} y}(a) \, h^i k^{3-i}$$

$$= \binom{3}{0} f_{yyy}(a) k^3 + \binom{3}{1} f_{yyx}(a) k^2 h + \binom{3}{2} f_{yxx}(a) k h^2$$

$$+ \binom{3}{3} f_{xxx}(a) h^3$$

$$\binom{3}{i} = \frac{3!}{i!(3-i)!} \qquad \binom{3}{0} = 1 \qquad \binom{3}{1} = \frac{3!}{2!} = 3$$

$$\binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

$$d^3 f_a(h,k) = f_{yyy}(a)\,k^3 + 3 f_{yyx}(a)\,k^2 h$$
$$+ 3 f_{yxx}(a)\,k h^2 + f_{xxx}(a)\,h^3$$

Ejemplo:

$$\boxed{f(x,y) = \log(1+2x+y)} \qquad a = (1,2)$$

$$df_{(1,2)}(h,k) = f_x(1,2)\,h + f_y(1,2)\,k$$

$$f(1,2) = \log(5)$$

$$f_x(x,y) = \frac{2}{1+2x+y}$$

$$f_y(x,y) = \frac{1}{1+2x+y}$$

$$\Rightarrow \quad df_{(1,2)}(h,k) = \frac{2}{1+2+2}h + \frac{1}{1+2+2}k$$

$$df_{(1,2)}(h,k) = \frac{2h}{5} + \frac{k}{5}$$

$$f(x,y) = f(1,2) + df_{(1,2)}(h,k) + \frac{1}{2!}d^2f_{(1,2)}(h,k) + \cdots$$

$x = 1+h$

$y = 2+k$

$$\cdots + \frac{1}{n!}d^nf_{(1,2)}(h,k) + r_k(h,k)$$

$$\lim_{h,k \to (0,0)} \frac{r_n(h,k)}{\|(h,k)\|^n} = 0$$

$$f_x(x,y) = \frac{2}{1+2x+y}$$

$$f_y(x,y) = \frac{1}{1+2x+y}$$

$$f_{xx}(x,y) = \frac{-2.2}{(1+2x+y)^2}$$

$$f_{xx}(1,2) = \frac{-4}{(1+2+2)^2} = \frac{-4}{25}$$

$$f_{xy}(x,y) = \frac{-1.2}{(1+2x+y)^2}$$

$$f_{xy}(1,2) = \frac{-2}{(1+2+2)^2} = \frac{-2}{25}$$

$$f_{yy}(x,y) = \frac{-1}{(1+2x+y)^2}$$

$$f_{yy}(1,2) = \frac{-1}{25}.$$

$$d^2 f_{(1,2)}(h,k) = f_{xx}(1,2)\,h^2 + 2 f_{xy}(1,2)\,hk + f_{yy}(1,2)\,k^2$$

$$= \frac{-4}{25}h^2 - \frac{4}{25}hk - \frac{k^2}{25}$$

$$\frac{1}{2!} d^2 f(1,2) = \frac{-2}{25}h^2 - \frac{2}{25}hk - \frac{k^2}{50}$$

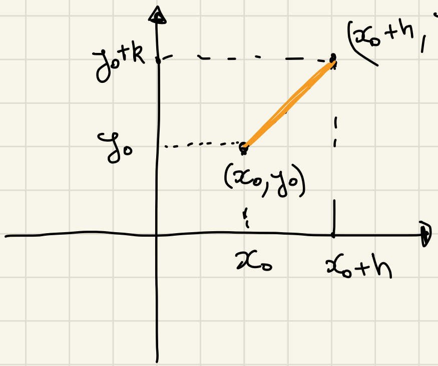
$$f(x,y) = \overbrace{\log(5)}^{f(1,2)} + \overbrace{\frac{2}{5}h + \frac{k}{5}}^{df_{(1,2)}(h,k)} + \overbrace{\frac{-2}{25}h^2 - \frac{2}{25}hk - \frac{k^2}{50}}^{\frac{1}{2} d^2 f_{(1,2)}(h,k)} + r_2(h,k)$$

$x = 1+h$

$y = 2+k$

$$\lim_{(h,k)\to(0,0)} \frac{r_2(h,k)}{\|(h,k)\|^2} = 0$$

Esquema de la prueba de Taylor.



$$t \longmapsto (x_0+th, y_0+tk)$$

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\alpha"] \arrow[rr, bend right, "g"'] & \mathbb{R}^2 \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$$g : [0,1] \longrightarrow \mathbb{R} \qquad g(t) = f(x_0+th, y_0+tk)$$

$$g'(t) = \langle \alpha'(t), \nabla f(x_0+th, y_0+tk) \rangle$$

$$= \langle (h,k), (f_x(x_0+th, y_0+tk), f_y(x_0+th, y_0+tk)) \rangle$$

$$= f_x(x_0+th, y_0+tk)h + f_y(x_0+th, y_0+tk)k$$

$$g'(0) = f_x(x_0,y_0)h + f_y(x_0,y_0)k.$$

Taylor de g en $t=0$. y orden n

$$g(t) = g(0) + g'(0)t + \frac{g''(0)t^2}{2} + \cdots + \frac{g^{(n)}(0)t^n}{n!}$$

$$+ r_n(t)$$

con $\lim_{t\to 0} \frac{r_n(t)}{t^n} = 0$

$$f(x_0+th, y_0+tk) = f(x_0,y_0) + \left(f_x(x_0,y_0)h + f_y(x_0,y_0)k\right)t$$

$$+ \cdots + \frac{df^n_{(x_0,y_0)}(h,k)\,t^n}{n!} +$$

Ejercicio: $g^{(n)}(0) = df^n_{(x_0,y_0)}(h,k)$

$\rightsquigarrow$ $t=1$

$$\Rightarrow \quad f(x_0+h, y_0+k) = f(x_0, y_0) + \underbrace{f_x(x_0,y_0)h + f_y(x_0,y_0)k}_{df_{(x_0,y_0)}(h,k)}$$

$$+ \cdots + df^n_{(x_0,y_0)}(h,k) + r_n(h,k)$$