Clase 33:

Derivadas de orden superior

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Eugenia Ellis

eellis@fing.edu.uy

Derivada de orden superior

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ diferenciable, podemos considerar

$f_x: \mathbb{R}^2 \longrightarrow \mathbb{R}$ y $f_y: \mathbb{R}^2 \longrightarrow \mathbb{R}$

podemos tomar las derivadas parciales de estas funciones

$a \in \mathbb{R}^2$

$$\frac{\partial^2 f}{\partial x \partial x}(a) = \left(\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x}\right)(a) = f_{xx}(a)$$

$$\frac{\partial^2 f}{\partial x \partial y}(a) = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}(a) = f_{xy}(a)$$

$$\frac{\partial^2 f}{\partial y \partial x}(a) = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}(a) = f_{yx}(a)$$

$$\frac{\partial^2 f}{\partial y \partial y}(a) = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}(a) = f_{yy}(a)$$

Ejemplo:

$$f(x,y) = x\,\text{sen}(y) + x^2y^2$$

Calculemos sus derivadas de orden 2.

$$f_x(x,y) = \text{sen}\,y + 2xy^2$$

$$f_y(x,y) = x\cos y + 2yx^2$$

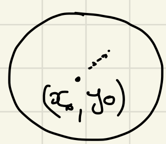
$$f_{xx}(x,y) = 2y^2$$

$$f_{xy}(x,y) = \frac{\partial f_y}{\partial x}(x,y) = \cos y + 4yx$$

$$\|$$

$$f_{yx}(x,y) = \frac{\partial f_x}{\partial y}(x,y) = \cos y + 4xy$$

$$f_{yy}(x,y) = -x\,\text{sen}\,y + 2x^2$$



Teorema de Schwarz

Si $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$, f_{xy} y f_{yx} existen en $B((x_0,y_0),\delta)$ y son continuas en (x_0,y_0)

$$\Longrightarrow \quad f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0).$$

Se dice que f es de clase C^2

Ejemplo: Veamos que si $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ no necesariamente ocurre que f es de clase C^2

$$f(x,y) = \begin{cases} xy^2 \operatorname{sen}\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

$$\boxed{f_x(x,y) = \begin{cases} y^2 \operatorname{sen}\frac{1}{y} & y \neq 0 \\ 0 & \end{cases}}$$

$$f_x(x_0, 0) = \lim_{h \to 0} \frac{f(x_0 + h, 0) - f(x_0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$$\boxed{f_y(x,y) = \begin{cases} x\left(2y\,\text{sen}\frac{1}{y} - \cos\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0. \end{cases}}$$

$$\frac{\partial}{\partial y}\left(x y^2\, \text{sen}\frac{1}{y}\right) = x\left(2y\,\text{sen}\frac{1}{y} + \cancel{y^2}\cdot\cos\frac{1}{y}\cdot\frac{-1}{\cancel{y^2}}\right)$$

$$= x\left(2y\,\text{sen}\frac{1}{y} - \cos\frac{1}{y}\right)$$

$$f_y(x_0,0) = \lim_{h\to 0}\frac{f(x_0,h) - f(x_0,0)}{h}$$

$$= \lim_{h\to 0}\frac{x_0 h^2\,\text{sen}\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0}\frac{\overbrace{x_0 h^{\cancel{2}}}^{\to 0}\,\overbrace{\text{sen}\frac{1}{h}}^{\text{acotada}}}{\cancel{h}} = 0$$

$$f_x(x,y) = \begin{cases} y^2 \operatorname{sen}\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

$$f_y(x,y) = \begin{cases} x\left(2y \operatorname{sen}\frac{1}{y} - \cos\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0. \end{cases}$$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{\cancel{2}} \cdot \operatorname{sen}\frac{1}{h} - 0}{\cancel{h}}$$

$$= \lim_{h \to 0} \underbrace{h}_{\to 0} \cdot \operatorname{sen}\frac{1}{h} = 0$$

acotado

$$\boxed{f_{xy}(0,0) = f_{yx}(0,0) = 0}$$

$$f_{xy}(x,y) = \begin{cases} 2y \operatorname{sen}\frac{1}{y} - \cos\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

$$f_{yx}(x,y) = \begin{cases} 2y \operatorname{sen}\frac{1}{y} + \cancel{y^2} \cos\frac{1}{y} \cdot \frac{-1}{\cancel{y^2}} & y \neq 0 \\ 0 & y = 0 \end{cases}.$$

$$\boxed{f_x(x,y) = \begin{cases} y^2 \operatorname{sen}\frac{1}{y} & y \neq 0 \\ 0 & y = 0 \end{cases}}$$

$$\boxed{f_y(x,y) = \begin{cases} x\left(2y \operatorname{sen}\frac{1}{y} - \cos\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases}.}$$

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y) \ \nexists$$

$\Rightarrow$ f_{xy} no es continua en $(0,0)$.

Ejemplo: Si existen $f_{xy}(x_0,y_0)$ y $f_{yx}(x_0,y_0)$ no necesariamente coinciden

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

$$f_x(x,y) = \frac{\frac{\partial\, xy(x^2-y^2)}{\partial x} \cdot (x^2+y^2) - 2x\, \frac{xy(x^2-y^2)}{x^2+y^2}}{(x^2+y^2)^2}$$

$(x,y) \neq (0,0).$

$$= \frac{\left(y(x^2-y^2) + xy\,2x\right)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{\left(yx^2 - y^3 + 2x^2y\right)(x^2+y^2) - 2x^4y + 2x^2y^3}{(x^2+y^2)^2}.$$

Comprobar que $f_{xy}(0,0) = -1$

$f_{yx}(0,0) = 1.$

Funciones a $\mathbb{R}^m$

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f(x_1, x_2, \ldots, x_n) = \left(f_1(x_1, \ldots, x_n), f_2(x_1, \ldots, x_n), \ldots f_m(x_1, \ldots, x_n)\right)$$

$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Ejemplo: $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$f(x,y,z) = \Big(\underbrace{e^z \operatorname{sen} y}_{f_1(x,y,z)}, \underbrace{x^2 y^2 e^{z^2}}_{f_2(x,y,z)} \Big)$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ es continua en a

$$\Longleftrightarrow$$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ tal que si

$$x \in B(a, \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$$

$$\|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| < \varepsilon$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ es diferenciable en a

$\iff$

Existe una transformación lineal $df_a : \mathbb{R}^n \to \mathbb{R}^m$ tal que

$$f(a+h) = f(a) + \boxed{df_a(h)} + r(h)$$

en donde $\displaystyle\lim_{h \to 0} \frac{r(h)}{\|h\|} = 0 \in \mathbb{R}^m$

$0 \in \mathbb{R}^n$

$$\begin{pmatrix} f_1(a+h) \\ \vdots \\ \vdots \\ f_m(a+h) \end{pmatrix} = \begin{pmatrix} f_1(a) \\ \vdots \\ f_m(a) \end{pmatrix} + \begin{pmatrix} \cdots J_1 \cdots \\ \hline \vdots \\ \hline J_m \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} + \begin{pmatrix} r_1(h) \\ \vdots \\ r_m(h) \end{pmatrix}$$

$$f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$f_i(a+h) = f_i(a) + J_i \cdot h + r_i(h)$$

$$J_i = \nabla f_i$$

$$df_a(h) = \begin{pmatrix} \nabla f_1(a) \\ \nabla f_2(a) \\ \vdots \\ \nabla f_m(a) \end{pmatrix} \cdot h.$$

$$\boxed{\underset{\substack{\text{Jacobiana} \\ \text{de } f \text{ en} \\ a}}{J_f(a) =} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \frac{\partial f_m}{\partial x_2}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}}$$

<u>Ejemplo:</u> $f(x,y,z) = (x^2y + e^z, \operatorname{sen} x + yz)$

$$Jf(x,y,z) = \begin{pmatrix} 2xy & x^2 & e^z \\ \cos x & z & y \end{pmatrix}$$

Teorema (Regla de la cadena II)

$$\left.\begin{array}{l} g: \mathbb{R}^k \longrightarrow \mathbb{R}^n \text{ diferenciable en } a \in \mathbb{R}^k \\ f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ diferenciable en } g(a) \in \mathbb{R}^n \end{array}\right\}$$

$\Rightarrow (f \circ g): \mathbb{R}^k \longrightarrow \mathbb{R}^m$ es diferenciable en $a \in \mathbb{R}^k$ y

$$J_{f \circ g}(a) = J_f(g(a)) \cdot J_g(a)$$

$$d(f \circ g)(a) = df_{g(a)} \circ dg_a$$