Clase 13:

Series - Criterio del equivalente y del cociente

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Criterio de equivalentes.

$\sum a_n$ y $\sum b_n$ series de términos positivos

a) • Si $\lim\limits_{n\to+\infty} \dfrac{a_n}{b_n} = L > 0 \Rightarrow$ las dos series convergen o divergen.

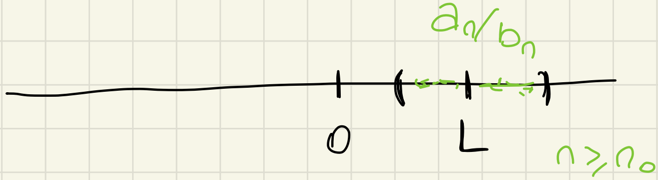
b) • Si $\lim\limits_{n\to+\infty} \dfrac{a_n}{b_n} = 0 \Rightarrow$ • $\sum b_n$ converge $\Rightarrow \sum a_n$ converge

• $\sum a_n$ diverge $\Rightarrow \sum b_n$ diverge

Dem: a)

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L > 0$$

Para $\varepsilon = \frac{L}{2}$, $\exists n_0 \in \mathbb{N}$ tal que



$$\frac{a_n}{b_n} \in E(L, L/2) \quad \forall n \geq n_0$$

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2} \quad \forall n \geq n_0$$

$$\Rightarrow \quad b_n \frac{L}{2} < a_n < \frac{3L}{2} b_n$$

$b_n > 0$

- Si $\sum a_n$ converge $\Rightarrow$ $\sum b_n$ converge

$$b_n < \frac{2}{L} a_n$$

Usamos criterio de comparación

- Si $\sum b_n$ converge $\Rightarrow$ $\sum a_n$ converge

$$a_n < \frac{3L b_n}{2}$$

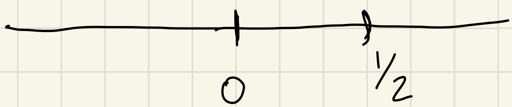
Criterio de comparación

$$\sum a_n \text{ converge} \iff \sum b_n \text{ converge.}$$

$a_n \geq 0$
$b_n \geq 0$

$$\sum a_n \underbrace{\text{no converge}}_{\text{diverge}} \iff \sum b_n \underbrace{\text{no converge}}_{\text{diverge}}$$

b)



Si $\lim\limits_{n\to+\infty} \dfrac{a_n}{b_n} = 0$

Para $\varepsilon = \dfrac{1}{2}$ $\exists n_0 \in \mathbb{N}$ tal que

$$0 \leq \frac{a_n}{b_n} < \frac{1}{2} \quad \forall n \geq n_0$$

$$\Rightarrow 0 \leq a_n < \frac{b_n}{2} \quad \forall n \geq n_0$$

$\Rightarrow$ • Si $\sum b_n$ converge $\Rightarrow$ $\sum a_n$ converge

criterio de comparación

$\sum a_n$ diverge $\Rightarrow$ $\sum b_n$ diverge

Ejemplos:

1) Vimos que $\sum \frac{1}{n(n+1)}$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\hookrightarrow S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$S_n = 1 \cancel{-\frac{1}{2}} \cancel{+\frac{1}{2}} \cancel{-\frac{1}{3}} \cancel{+\frac{1}{3}} \cancel{-\frac{1}{4}} \cancel{+} \cdots \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \qquad \lim_{n \to +\infty} S_n = 1$$

$\Rightarrow \sum \frac{1}{n(n+1)}$ converge

$$\frac{1}{n(n+1)} \sim \frac{1}{n^2} \qquad \lim_{n\to+\infty} \frac{\frac{1}{n(n+1)}}{1/n^2} = 1$$

$\Rightarrow$ $\sum \frac{1}{n^2}$ converge

criterio del equivalente

2) $\sum \frac{1}{n^\alpha}$ $\alpha > 2$

$\alpha > 2$ $n^2 < n^\alpha \Rightarrow \frac{1}{n^\alpha} < \frac{1}{n^2}$

como $\sum \frac{1}{n^2}$ converge

$\Rightarrow$ criterio de comparación $\Rightarrow \sum \frac{1}{n^\alpha}$ converge

$\forall \alpha > 2$

3) $\sum \operatorname{sen}\left(\frac{1}{n}\right)$

$\operatorname{sen}\left(\frac{1}{n}\right) \sim \frac{1}{n} \Rightarrow$ criterio de equivalentes

$\sum \operatorname{sen}\left(\frac{1}{n}\right)$ y $\sum \frac{1}{n}$ tienen el mismo comportamiento.

$\sum \frac{1}{n}$ diverge $\qquad \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$

$\Rightarrow \sum \operatorname{sen}\left(\frac{1}{n}\right)$ diverge.

4) $\sum \frac{1}{\sqrt{n(n+2)}}$ diverge porque.

$\frac{1}{\sqrt{n(n+2)}} \sim \frac{1}{n}$ y $\sum \frac{1}{n}$ diverge

Criterio del cociente

criterio de D'Alambert

$\sum a_n$ serie de términos positivos

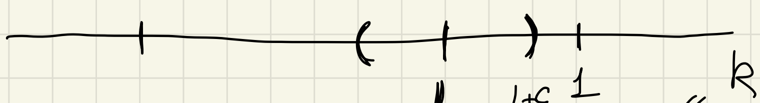
tal que $\boxed{\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L}$

$\Rightarrow$

- Si $L < 1 \Rightarrow \sum a_n$ converge
- Si $L > 1 \Rightarrow \sum a_n$ diverge.

Dem:

- $$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = L < 1$$



Consideramos $\varepsilon > 0$ tq $\underset{k}{\underbrace{L+\varepsilon}} < 1$,

$\exists n_0 \in \mathbb{N}$ tq $\frac{a_{n+1}}{a_n} \leq \underbrace{L+\varepsilon}_{k} < 1$

$\forall n \geq n_0$

$$\frac{a_{n+1}}{a_n} \leq k < 1 \quad \forall n \geq n_0$$

$$\Rightarrow \quad a_{n+1} \leq a_n k \quad \forall n \geq n_0$$

$$\Rightarrow a_{n+1} \leq k a_n \leq k^2 a_{n-1} \leq \cdots \leq k^{n+1-n_0} a_{n_0}$$

$$\Rightarrow \quad a_n \leq k^{n-n_0} a_{n_0} \quad \forall n > n_0$$

$$= k^n \cdot \boxed{\frac{a_{n_0}}{k^{n_0}}} \in \mathbb{R} = \lambda$$

$$a_n \leq k^n \cdot \lambda$$

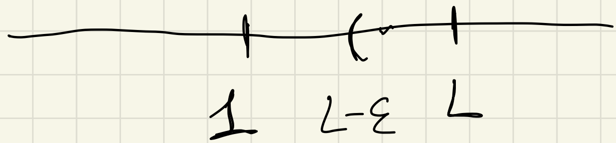
$\sum k^n$ converge $\Rightarrow$ $\sum k^n \lambda$ converge

$k < 1$

$\Rightarrow$ $\sum a_n$ converge

criterio de comparación

Si $L>1$ Sea $\varepsilon>0$ tq $1<\underbrace{L-\varepsilon}_{k}$

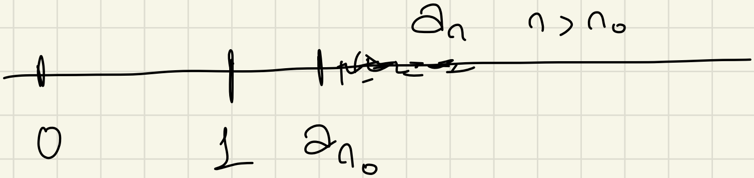


$$\Rightarrow 1<k<\frac{a_{n+1}}{a_n} \quad \forall n \geq n_0$$

$$0 < a_{n_0} < a_{n_0}k < a_{n+1} \quad \forall n \geq n_0$$

$\in \mathbb{R}^+$

$$\Rightarrow a_{n_0} < a_{n+1} \quad \forall n \geq n_0.$$



$$\lim_{n\to+\infty} a_n \neq 0 \Rightarrow \sum a_n$$

no converge

$\Rightarrow \quad a_n \geq 0 \quad \sum a_n$ diverge.

Ejemplos.

1) $\sum \frac{n}{n+3}$ es una serie de términos positivos

$\Rightarrow$ Converge ó Diverge

Si $\sum \frac{n}{n+3}$ fuere convergente

$\Longrightarrow \qquad \lim \frac{n}{n+3} = 0$

condición necesaria

pero $\lim\limits_{n \to \infty} \frac{n}{n+3} = 1$

$\Rightarrow \qquad \sum \frac{n}{n+3}$ diverge

2) $\sum \frac{n!}{n^n}$ $\qquad a_n = \frac{n!}{n^n}$

Usemos el criterio del cociente.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!}$$

$$= \frac{\cancel{(n+1)}!}{\cancel{n!}} \frac{n^n}{(n+1)^n \cdot \cancel{(n+1)}}$$

$$= \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{\frac{n+1}{n}}\right)^n$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e} = L < 1$$

$\Rightarrow$ $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge

criterio del cociente

2) Ejercicio clasificar

$$\sum \frac{2^n}{n!}$$

3) Aplicar el criterio del cociente a

$$\sum \frac{1}{n} \text{ y } \sum \frac{1}{n^2}.$$