Clase 35

Fracciones simples:

① $\dfrac{1}{(x-\alpha)^m}$ con $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}^+$

② $\dfrac{1}{(x^2+bx+c)^m}$ con $\Delta = b^2-4c<0$, $m \in \mathbb{Z}^+$

③ $\dfrac{x}{(x^2+bx+c)^m}$ con $\Delta = b^2-4c<0$, $m \in \mathbb{Z}^+$

<u>Obs.</u> ② y ③ se estudian simultaneamente considerando $\dfrac{A+Bx}{(x^2+bx+c)^m}$

<u>Recordar:</u>

$$\int x^{-1}\,dx = \log|x| + C$$

$$\int \frac{1}{1+x^2}\,dx = \operatorname{arctg}(x) + C$$

Como integrar fracciones simples:

(1) $$\int \frac{1}{(x-\alpha)^m}\,dx = \int (x-\alpha)^{-m}\,dx$$

$$= \begin{cases} \log|x-\alpha| + C & \text{si } m=1 \\[2mm] \dfrac{(x-\alpha)^{-m+1}}{-m+1} & \text{si } m\neq 1 \end{cases}$$

Para (2) y (3):

Obs. $$\int \frac{A+Bx}{(x^2+bx+c)^m}\,dx = \,?$$ $\Delta = b^2-4c<0$

$\Delta = -p^2$ con $p>0$

$$\frac{4^m}{4^m}\int \frac{A+Bx}{(x^2+bx+c)^m}\,dx = 4^m\int \frac{A+Bx}{(4x^2+4bx+4c)^m}\,dx$$

$$4^m\cdot(x^2+bx+c)^m = (4x^2+4bx+4c)^m$$

Completamos cuadrados: $4x^2+4bx+4c =$

$$\underbrace{4x^2+4bx+b^2}_{cp}-b^2+4c =$$

$$(2x+b)^2 - \Delta =$$

$$(2x+b)^2 + p^2 =$$

$$p^2\left[\left(\frac{2x+b}{p}\right)^2 + 1\right]$$

$$= 4^m \int \frac{A+Bx}{p^{2m}\left[\left(\frac{2x+b}{p}\right)^2+1\right]^m}\,dx$$

$$= \frac{4^m}{p^{2m}} \int \frac{A+Bx}{\left[\left(\frac{2x+b}{p}\right)^2+1\right]^m}\,dx$$

$C =$

Cambio de variable: $u = \frac{2x+b}{p}$ "despejo"

$$du = \frac{2}{p}\,dx \rightarrow dx = \frac{p}{2}\,du$$

$$2x+b = p \cdot u$$

$$2x = p \cdot u - b$$

$$x = \frac{p}{2} \cdot u - \frac{b}{2}$$

$$\Rightarrow A+Bx =$$

$$A + B\left(\frac{p}{2}u - \frac{b}{2}\right)$$

$$\frac{Bp}{2} \cdot u + \left(A - \frac{bB}{2}\right)$$

$$B' = \frac{Bp}{2} \qquad A' = A - \frac{bB}{2}$$

$$= C \cdot \int \frac{A' + B'u}{(u^2+1)^m} \, du$$

$$= C \cdot A' \cdot \underbrace{\int \frac{1}{(u^2+1)^m} \, du}_{*} + C B' \underbrace{\int \frac{u}{(u^2+1)^m} \, du}_{**}$$

$$** = \int \frac{u}{(u^2+1)^m} \, du = \int \frac{1}{v^m} \frac{1}{2} \, dv$$

$$= \frac{1}{2} \int v^{-m} \, dv$$

$$\boxed{\begin{aligned} v &= u^2+1 \\ dv &= 2u \, du \end{aligned}}$$

Si $m = 1$: $** = \frac{1}{2} \log|v|$

$$= \frac{1}{2} \log(u^2+1)$$

$$= \frac{1}{2} \log\left(\left(\frac{2x+b}{\mu}\right)^2 + 1\right)$$

Si $m \neq 1$: $(**) = \dfrac{1}{2} \cdot \dfrac{v^{-m+1}}{-m+1}$

$$= \frac{1}{2(1-m)} \frac{1}{v^{m-1}}$$

$$= \frac{1}{2(1-m)} \cdot \frac{1}{(u^2+1)^{m-1}}$$

$$= \frac{1}{2(1-m)} \cdot \frac{1}{\left(\left(\frac{2x+b}{\mu}\right)^2+1\right)^{m-1}}$$

La parte $(*) = \displaystyle\int \frac{1}{(u^2+1)^m} du$ se resuelve por recurrencia usando integración por parte de la siguiente forma:

$$I_m = \int \frac{1}{(u^2+1)^m} du$$

$$I_1 = \int \frac{1}{u^2+1} du = \operatorname{arctg}(u)$$

Si $m \geq 1$:

$$I_m = \int \frac{1}{(u^2+1)^m}\,du = \int \overset{\text{"int"}}{1}\cdot\overset{\text{"der"}}{(u^2+1)^{-m}}\,du$$

$$= u\,(u^2+1)^{-m} - \int u\cdot(-m)(u^2+1)^{-m-1}\cdot 2u\,du$$

$$= \frac{u}{(u^2+1)^m} + 2m \int \frac{u^2+1-1}{(u^2+1)^{m+1}}\,du$$

$$= \frac{u}{(u^2+1)^m} + 2m \underbrace{\int \frac{1}{(u^2+1)^m}\,du}_{= I_m} - 2m \underbrace{\int \frac{du}{(u^2+1)^{m+1}}}_{= I_{m+1}}$$

$$\Rightarrow\ I_m = \frac{u}{(u^2+1)^m} + 2m\,I_m - 2m\,I_{m+1}$$

$$\Rightarrow\ 2m\,I_{m+1} = \frac{u}{(u^2+1)^m} + (2m-1)\,I_m$$

$$\boxed{I_{m+1} = \frac{u}{2m(u^2+1)^m} + \frac{2m-1}{2m} I_m \quad \forall m \geq 1}$$

Ejemplo: $\displaystyle\int \frac{1}{(1+u^2)^2}\,du = I_2$

$$I_2 = \frac{u}{2(u^2+1)} + \frac{1}{2} I_1$$

$$= \frac{u}{2(u^2+1)} + \frac{1}{2}\operatorname{arctg}(u)$$

$$\boxed{\int \frac{1}{(1+u^2)^2}\,du = \frac{u}{2(u^2+1)} + \frac{1}{2}\operatorname{arctg}(u)+C}$$

Va: $\displaystyle\frac{2(u^2+1)-4u^2}{4(u^2+1)^2} + \frac{1}{2(u^2+1)}$

$$\frac{1}{2(u^2+1)} - \frac{u^2}{(u^2+1)^2} + \frac{1}{2(u^2+1)}$$

$$\frac{1}{u^2+1} - \frac{u^2}{(u^2+1)^2} = \frac{\cancel{u^2}+1-\cancel{u^2}}{(u^2+1)^2} \quad \checkmark$$

→ Sabemos integrar fracciones simples ✓

→ Falta ver como expresar las funciones racionales propias como combinación lineal de fracciones simples.

<u>Teo.</u> Si $F(x) = \frac{P(x)}{Q(x)}$ es un fración racional <u>propia</u> entonces

$F(x)$ se puede expresar como combinación lineal de fracciones simples.

Más aun:

1. Si $(x-\alpha)^m$ divide a $Q(x)$ entonces

en la combinación lineal aparece

$$\frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_m}{(x-\alpha)^m}$$

donde $A_1, A_2, \ldots, A_m$ son reales a determinar

2. Si $(x^2+bx+c)^m$ divide a $Q(x)$ con $\Delta = b^2-4c < 0$ entonces en la combinación lineal aparece

$$\frac{A_1+B_1x}{x^2+bx+c} + \frac{A_2+B_2x}{(x^2+bx+c)^2} + \cdots + \frac{A_m+B_mx}{(x^2+bx+c)^m}$$

donde $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$ son reales a determinar.

Ejemplos concretos con casos especiales:

Obs: El teorema anterior nos asegura que $f(x) = \dfrac{x^2+3x-1}{(x-2)(x-3)(x-5)^2(x^2+x+1)^2}$

puede escribirse como:

$$f(x) = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x-5} + \frac{D}{(x-5)^2} +$$

$$\frac{E}{x^2+x+1} + \frac{F}{(x^2+x+1)^2}$$

con $A, B, C, D, E, F \in \mathbb{R}$

$$\Rightarrow \int f(t)dt = A\int \frac{1}{t-2} + B\int \frac{1}{t-3} + C\int \frac{1}{t-5} + \dots$$

Ejemplo: Q con raíces reales distintas.

(1) $$\int \frac{x+1}{x^2-3x+2}\,dx$$

$$x^2-3x+2=0 \quad \begin{cases} \dfrac{3+1}{2}=2 \\[2ex] \dfrac{3-1}{2}=1 \end{cases}$$

$$\Delta = 9-8=1$$

$$\frac{x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$\begin{aligned} x+1 &= A(x-1)+B(x-2) \\ &= Ax - A + Bx - 2B \\ &= (A+B)x + (-A-2B) \end{aligned}$$

$$\begin{cases} A+B=1 \\ -A-2B=1 \end{cases} \rightarrow \begin{cases} A+B=1 \\ -B=2 \end{cases}$$

$$\boxed{\begin{matrix} B=-2 \\ A=3 \end{matrix}}$$

$$\int \frac{x+1}{x^2-3x+2}\,dx = \int\left(\frac{3}{x-2} + \frac{-2}{x-1}\right)dx$$

$$= 3\int \frac{1}{x-2}\,dx - 2\int \frac{1}{x-1}\,dx$$

$$= 3\log|x-2| - 2\log|x-1| + C$$

$$\boxed{\int \frac{(x+1)\,dx}{x^2-3x+2} = \log\left(\frac{|x-2|^3}{|x-1|^2}\right) + C}$$

(2) $$\int \frac{x^2+x+1}{(x-1)(x-2)(x-3)}\,dx$$

$$\boxed{\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}}$$

teo.

<u>1º Pas</u>: Multiplicar todo por $(x-1)(x-2)(x-3)$:

$$x^2+x+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$$= (A+B+C)x^2 + (-5A-4B-3C)x + (6A+3B+2C)$$

$$\Rightarrow \begin{cases} A+B+C=1 \\ -5A-4B-3C=1 \\ 6A+3B+2C=1 \end{cases} \Rightarrow A= \ , B= \ , C=$$

2° Paso: $\dfrac{x^2+x+1}{(x-2)(x-3)} = A + \left(\dfrac{B}{x-2}+\dfrac{C}{x-3}\right)(x-1)$

$$\xrightarrow[x=1]{} \frac{3}{(-1)(-2)} = A \rightarrow \boxed{A = 3/2}$$

$$\frac{x^2+x+1}{(x-1)(x-3)} = B + (\sim)(x-2)$$

$$\xrightarrow[x=2]{} \frac{7}{(-1)} = B \rightarrow \boxed{B=-7}$$

$$\frac{x^2+x+1}{(x-1)(x-2)} = C + (\sim)(x-3)$$

$$\xrightarrow[x=3]{} \frac{13}{2\cdot 1} = C \rightarrow \boxed{C = \frac{13}{2}}$$

$$\int \frac{x^2+x+1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{3/2}{x-1} + \int \frac{-7}{x-2} + \int \frac{13/2}{x-3}$$

$$\boxed{\int = \frac{3}{2} \cdot \log|x-1| - 7\log|x-2| + \frac{13}{2}\log|x-3| + C}$$