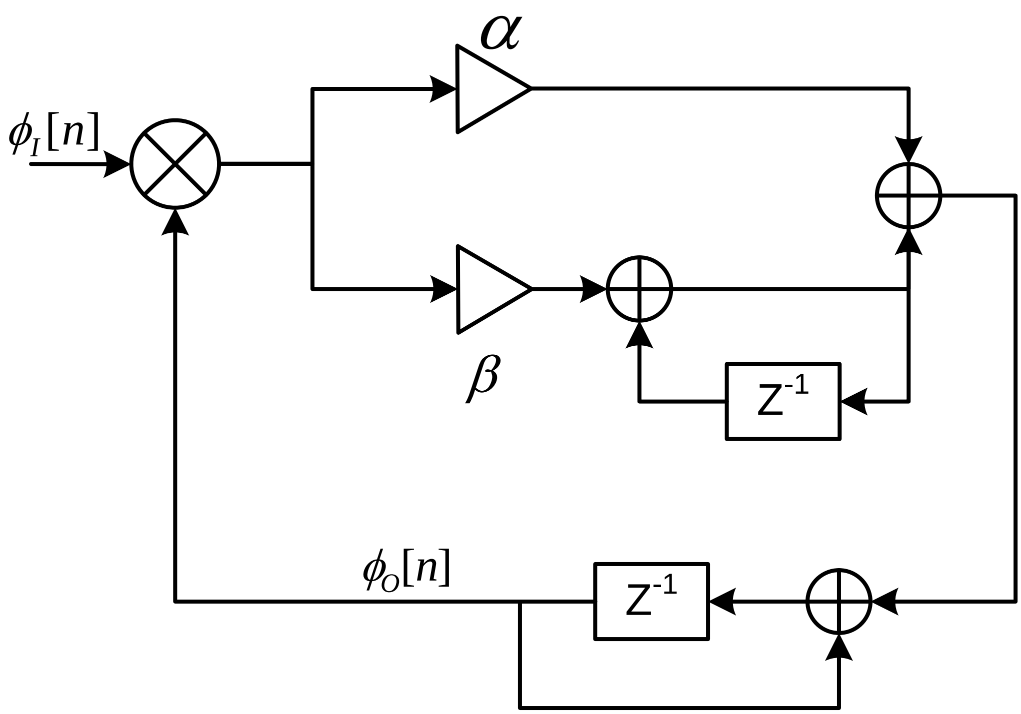

In the figure of the control loop above, we want to find the transfer function, so we solve for:

$$H[z] = \frac{\Phi_O[z]}{\Phi_I[z]}$$

In the time domain representation, we have:

$$\phi_O[n] = \phi_O[n-1] + \alpha\big[\phi_I[n] - \phi_O[n]\big] + \sum_{i=0}^{n} \beta\big[\phi_I[n-i] - \phi_O[n-i]\big]$$

The last integral is a first-order filter, which is easily shown to be represented in the z-domain as:

$$\frac{z}{z-1}$$

So using the bilinear z-transform, we convert the discrete time series to:

$$\Phi_O[z] = \Phi_O[z]z^{-1} + \alpha\big[\Phi_I[z] - \Phi_O[z]\big] + \ \beta\big[\Phi_I[z] - \Phi_O[z]\big]\frac{z}{z-1}$$

Rearranging, weget:

$$\Phi_O[z]\left[1 - z^{-1} + \alpha + \beta\frac{z}{z-1}\right] = \Phi_I[z]\left[\alpha + \beta\frac{z}{z-1}\right]$$

Multiplying both sides by $\frac{z}{z-1}$ gives us:

$$\Phi_O[z]\left[\frac{z-1}{z-1}+\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}\right]=\Phi_I[z]\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}$$

$$\Phi_O[z]\left[1+\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}\right]=\Phi_I[z]\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}$$

Therefore,

$$H[z]=\frac{\Phi_O[z]}{\Phi_I[z]}=\frac{\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}}{1+\left[\alpha+\beta\frac{z}{z-1}\right]\frac{z}{z-1}}$$

We want to reformat this equation into the classical 2[nd] loop function:

$$H_{REF}[s]=\frac{\varpi_n^2}{s^2+2\zeta\varpi_n s+\varpi_n^2}$$

Using Tustin's method to move $H_{REF}[s]$ to the z-domain:

$$s=\frac{2}{T_s}\frac{Z-1}{Z+1}$$

If we work through the algebra, and substitute $\theta_n=\frac{\varpi_n T_s}{2}$, where θ_n is the undamped natural frequency, we get:

$$H_{REF}[z]=\frac{4\theta_n(\zeta+\theta_n)}{1+2\zeta\theta_n+\theta_n^2}\frac{z-\frac{\zeta}{\zeta+\theta_n}}{z^2-2\frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}z+\frac{1-2\zeta\theta_n+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}}$$

This looks messy now, but we can find substitutions for this equation in terms of our loop gains, α and β, where α is known as the proportional gain and β is known as the integral gain. Specifically,

$$\frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}=1-\frac{\alpha+\beta}{2}$$

$$\frac{\alpha+\beta}{2}=1-\frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}=\frac{1+2\zeta\theta_n+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}-\frac{1+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}=\frac{2\zeta\theta_n+2\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}$$

$$\alpha+\beta=\frac{4\zeta\theta_n+4\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}$$

Similarly,

$$\frac{1-2\zeta\theta_n+\theta_n^2}{1+2\zeta\theta_n+\theta_n^2}=1-\alpha$$

$$\alpha = 1 - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} = \frac{1 + 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} - \frac{1 - 2\zeta\theta_n + \theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2} =$$

$$\alpha = \frac{4\zeta\theta_n}{1 + 2\zeta\theta_n + \theta_n^2}$$

Which leaves,

$$\alpha = \frac{4\zeta\theta_n}{1 + 2\zeta\theta_n + \theta_n^2}$$

$$\beta = \frac{4\theta_n^2}{1 + 2\zeta\theta_n + \theta_n^2}$$

$$\theta_n = \frac{\varpi_n T_s}{2}$$