

Señales y Sistemas

Transformada de Fourier de variable continua
(CTFT)

Instituto de Ingeniería Eléctrica

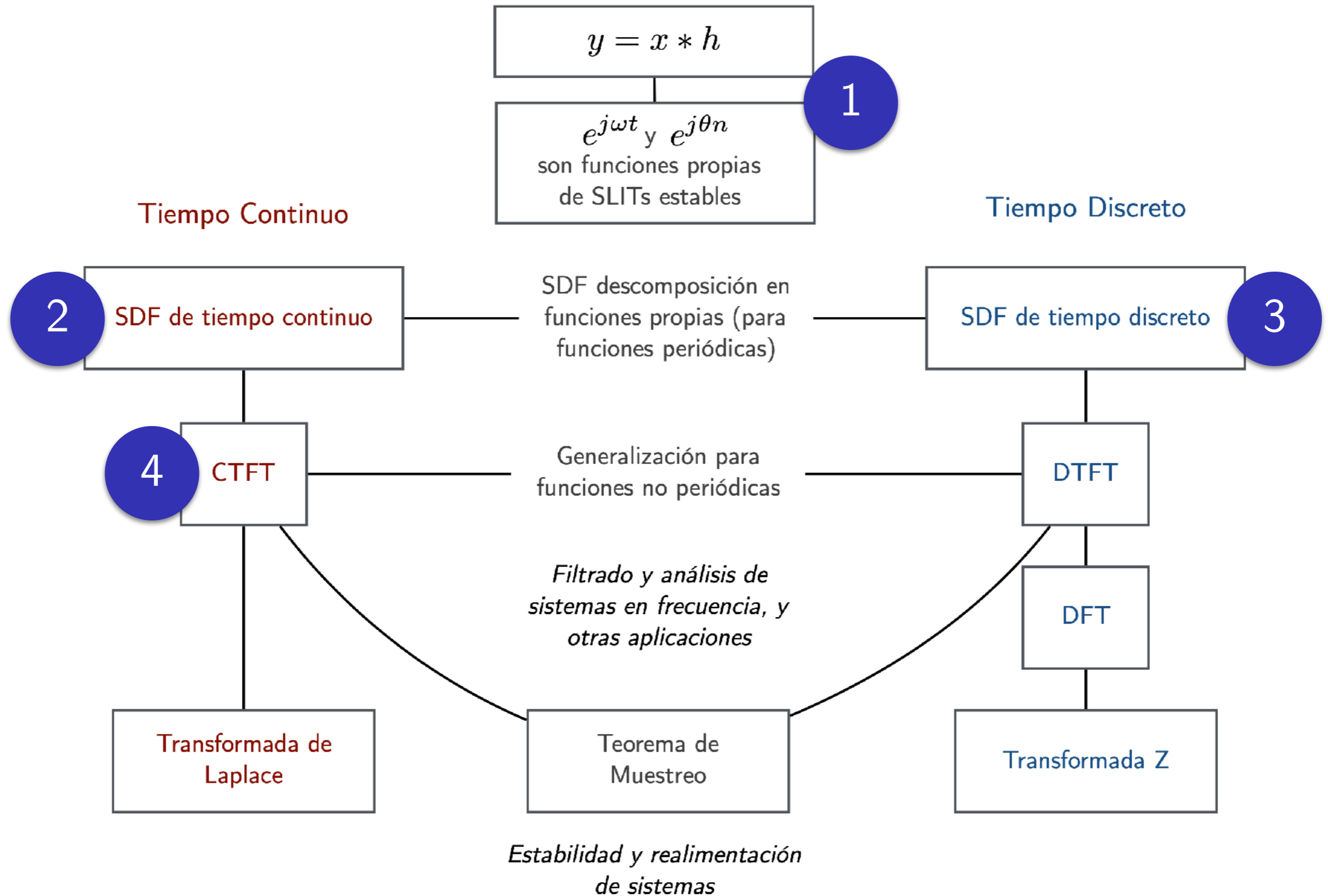


FACULTAD DE
INGENIERÍA



UNIVERSIDAD
DE LA REPÚBLICA
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Señales y sistemas



* tiempo o variable

Series de Fourier

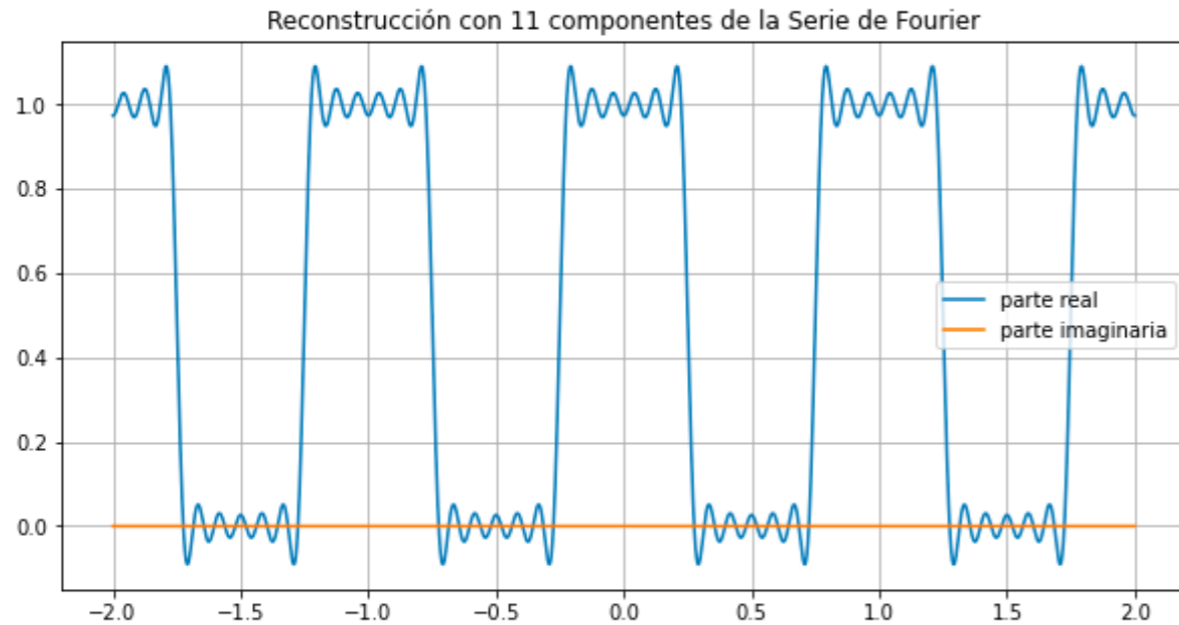
- $x(t)$ **periódica**, de período T : $x(t + T) = x(t) \forall t$ ($\omega_0 = 2\pi/T$)

Serie de Fourier

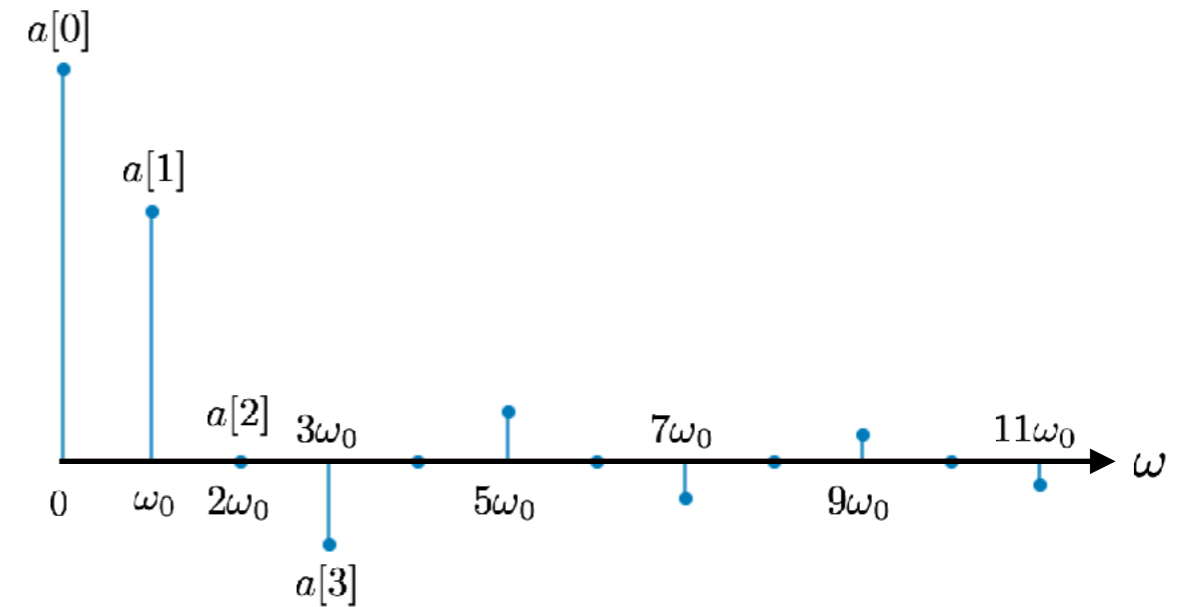
$$x(t) = \sum_{k=-\infty}^{+\infty} a[k] e^{jk\omega_0 t} \quad \text{Síntesis}$$

$$a[k] = \frac{1}{T} \int_{\langle T \rangle} x(\tau) e^{-jk\omega_0 \tau} d\tau \quad \text{Análisis}$$

$x(t)$

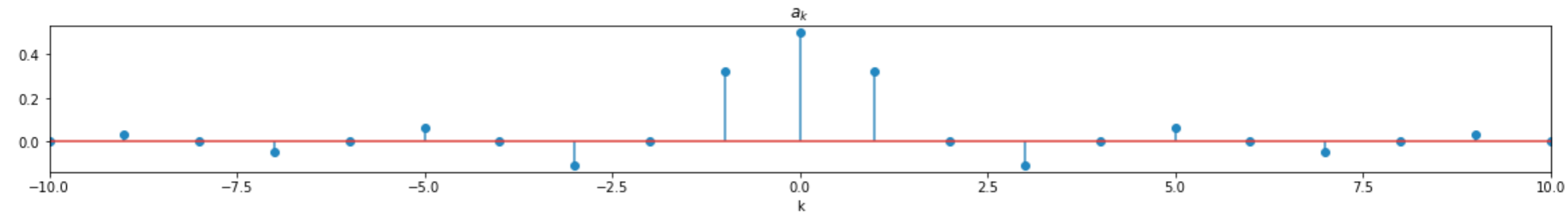
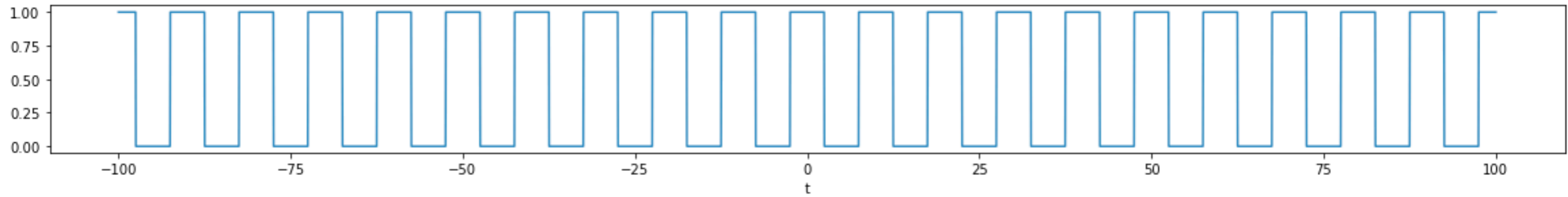


$X(j\omega)$



Series de Fourier

Onda cuadrada $x(t)$, $T = 10$

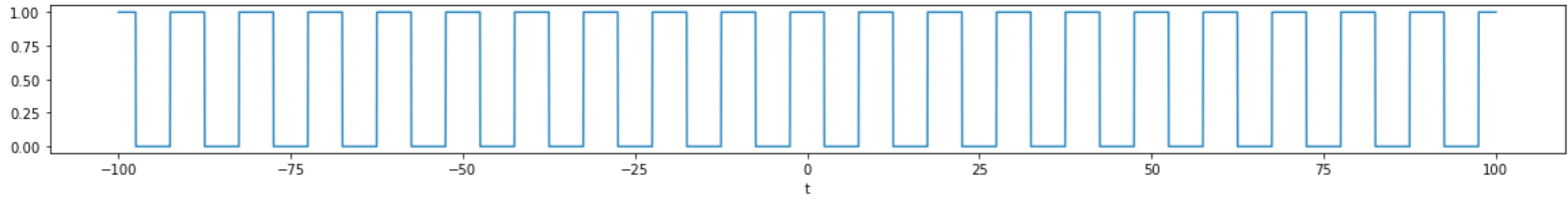


$$x(t) = \sum_k \Pi\left(\frac{t - kT}{2t_0}\right) \quad a_k = \frac{1}{k\pi} \sin(\omega_k t_0), \quad \omega_k = k\omega_0 \quad T \uparrow \Rightarrow \omega_0 \downarrow$$

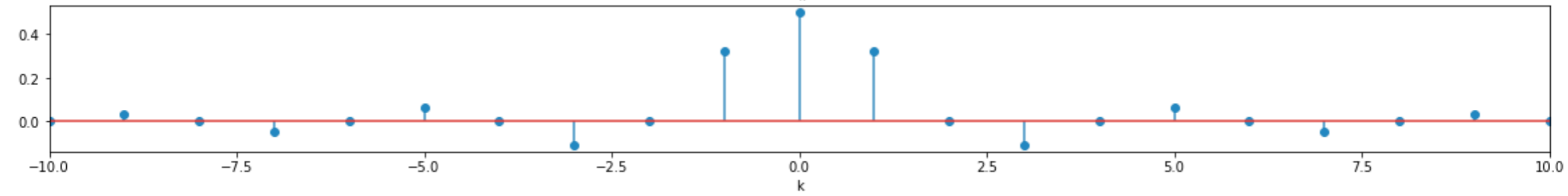
$$T a_k = \frac{2}{\omega_k} \sin(\omega_k t_0)$$

Series de Fourier

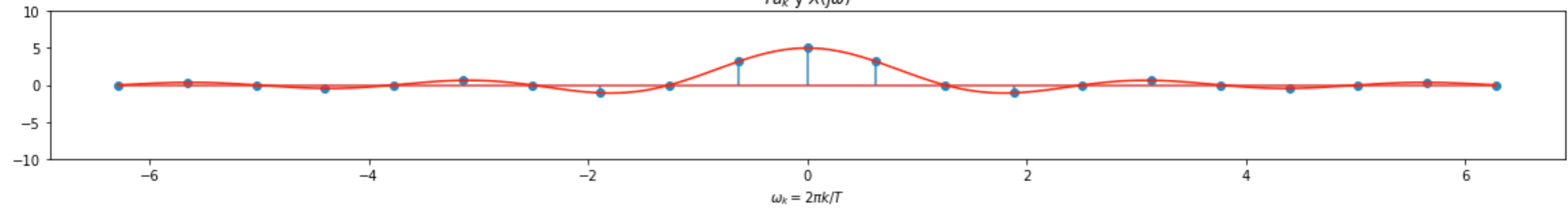
Onda cuadrada $x(t)$, $T = 10$



a_k

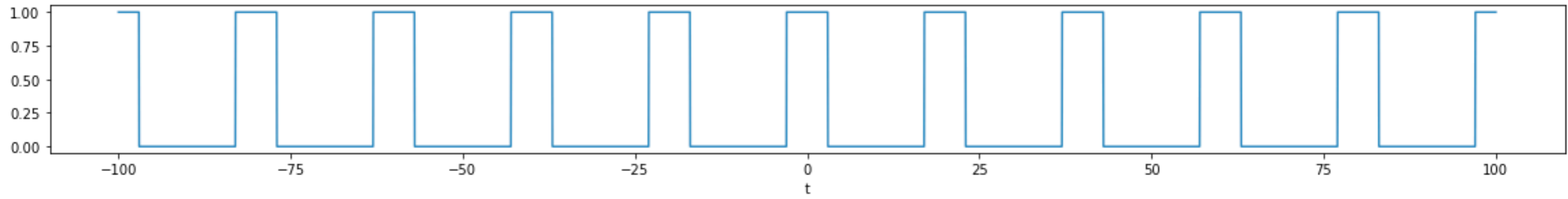


Ta_k y $X(j\omega)$

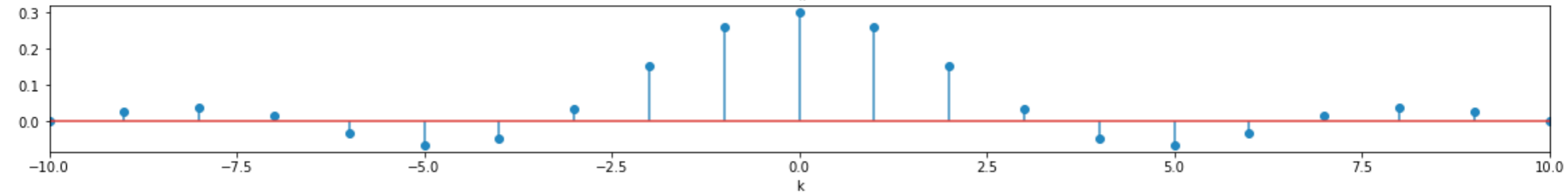


Series de Fourier

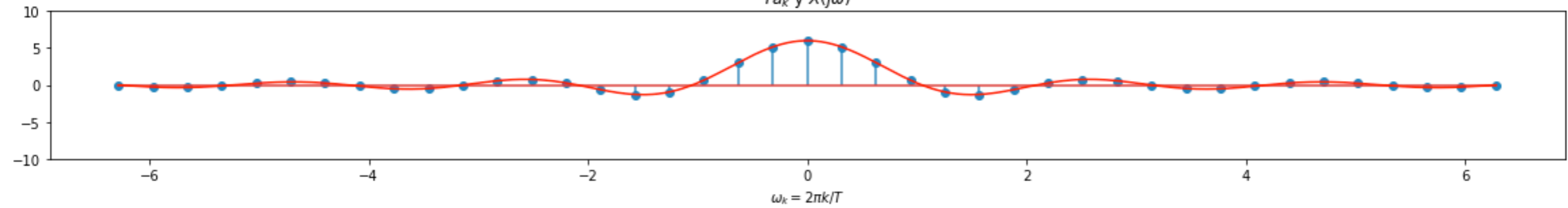
Onda cuadrada $x(t)$, $T = 20$



a_k

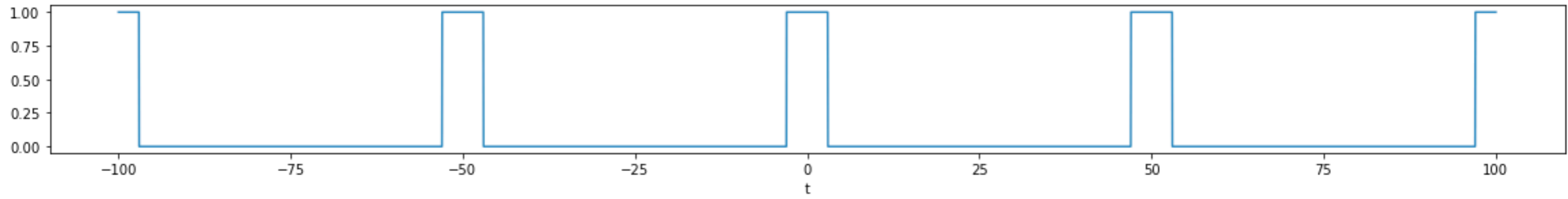


Ta_k y $X(j\omega)$

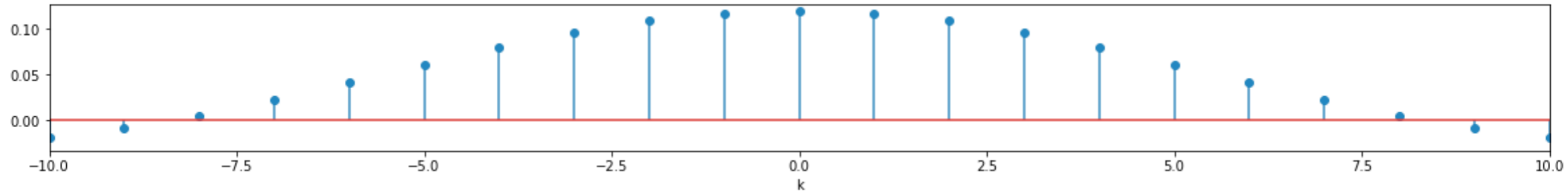


Series de Fourier

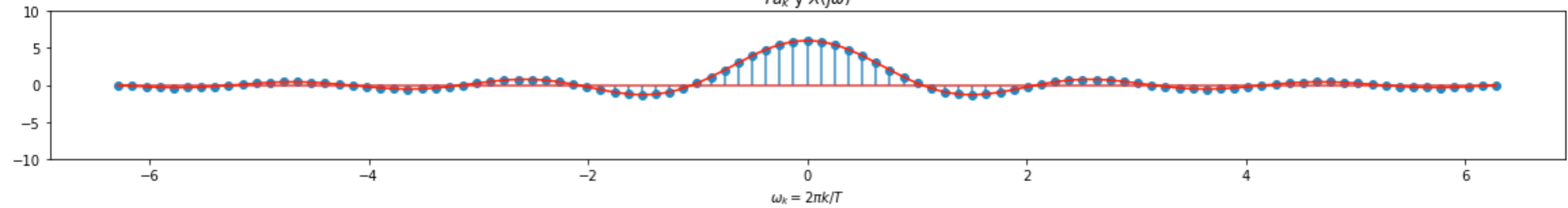
Onda cuadrada $x(t)$, $T = 50$



a_k

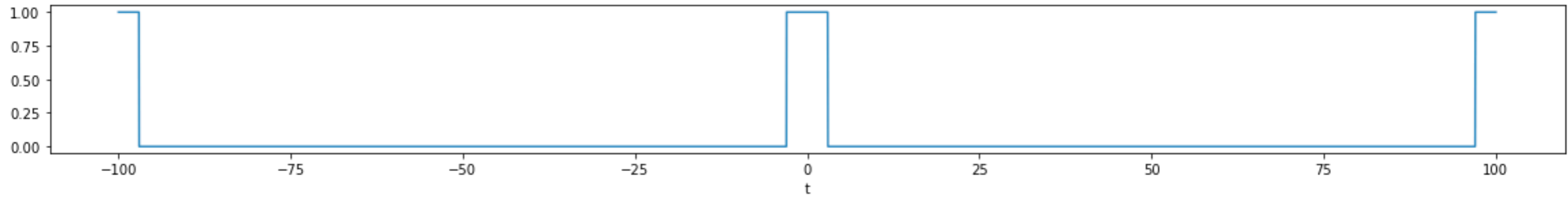


Ta_k y $X(j\omega)$

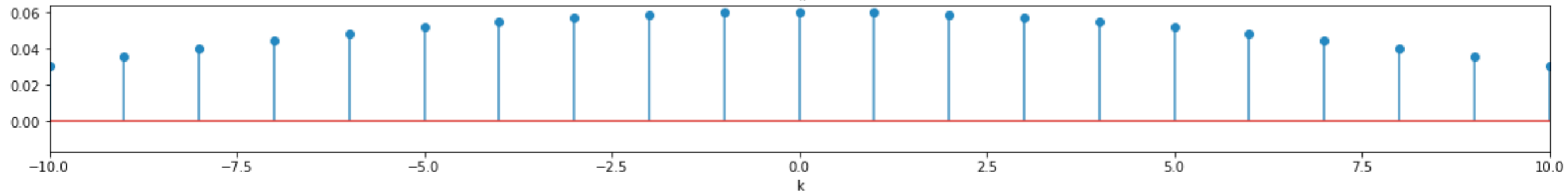


Series de Fourier

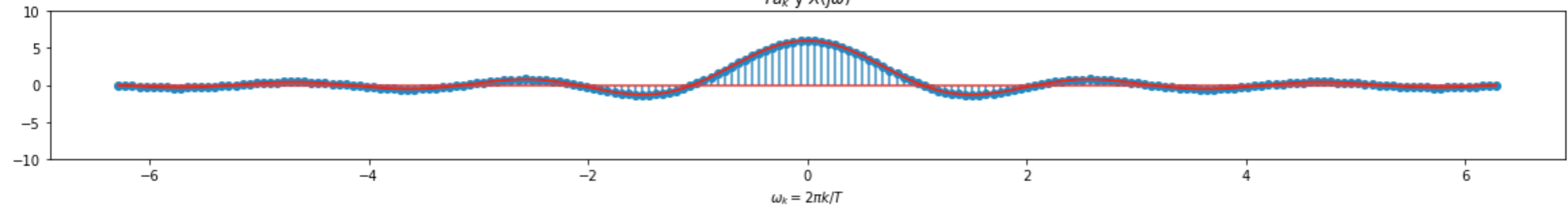
Onda cuadrada $x(t)$, $T = 100$



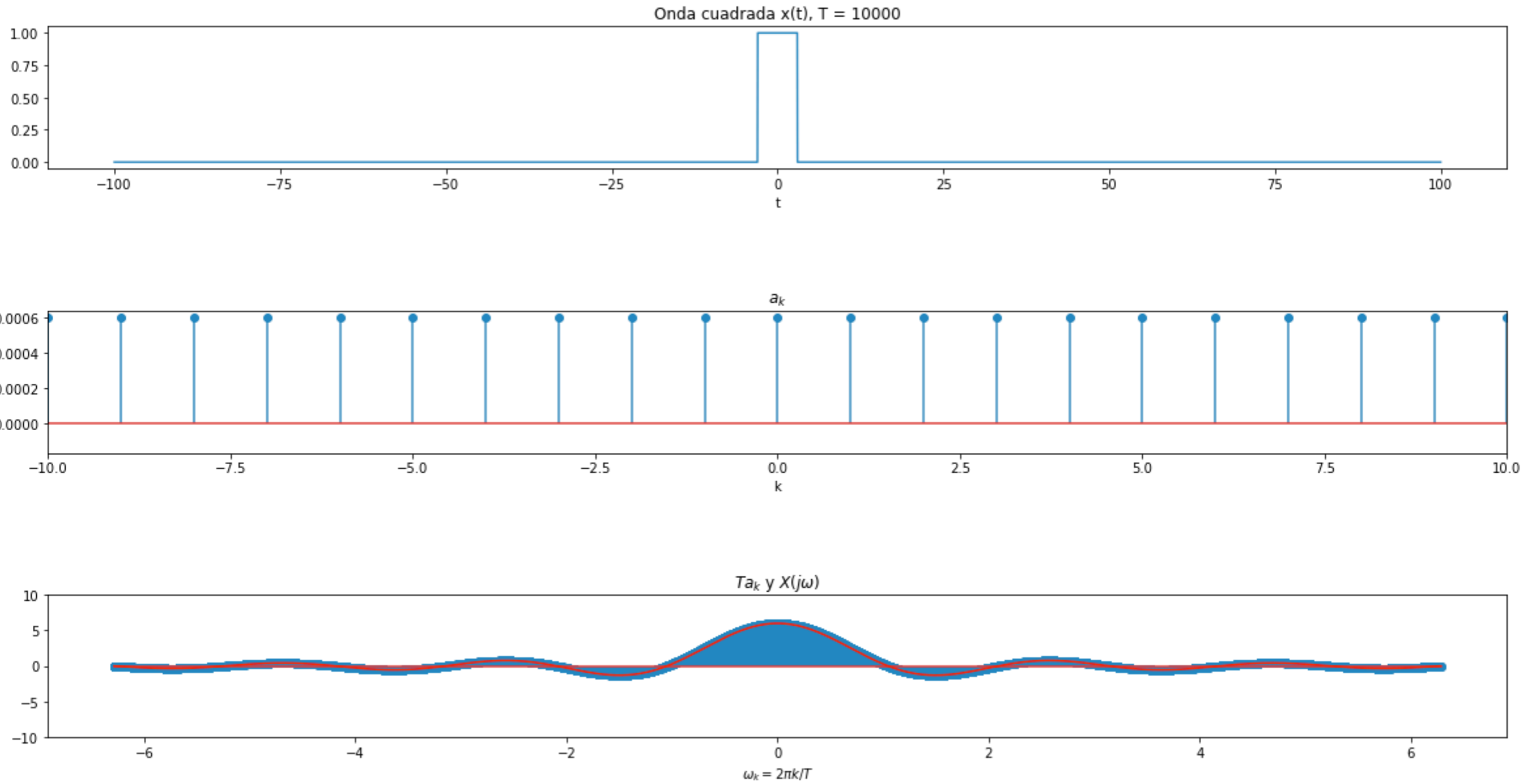
a_k



Ta_k y $X(j\omega)$



Series de Fourier



De la Serie de Fourier a la CTFT

$$\begin{aligned}x(t) &= \frac{1}{T} \sum_k T a_k e^{j\omega_0 k t} = \frac{1}{T} \sum_k X(j\omega_k) e^{j\omega_k t} \quad \omega_k = \omega_0 k \\ &= \frac{1}{2\pi} \sum_k X(j\omega_k) e^{j\omega_k t} \Delta\omega_k \quad \Delta\omega_k = \frac{2\pi}{T} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega \quad T \uparrow \infty, \Delta\omega_k \rightarrow 0\end{aligned}$$

Transformada de Fourier de variable continua (CTFT)

- $X(j\omega)$ es la TF de $x(t)$ (no es necesario que sea periódica)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau \quad \text{Análisis}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Síntesis}$$

- La relación entre la CTFT y la SF es estrecha.
- ¿Cuál es la CTFT de una señal periódica?
- Convergencia de la CTFT: son las condiciones de Dirichlet
 1. $x(t)$ es absolutamente integrable,
 2. en cada intervalo finito hay finitos máximos o mínimos,
 3. en cada intervalo finito hay finitas discontinuidades finitas.
- 2 y 3 son necesarias para que se cumpla la ecuación de análisis en todo t excepto en las discontinuidades (recordar fenómeno de Gibbs).

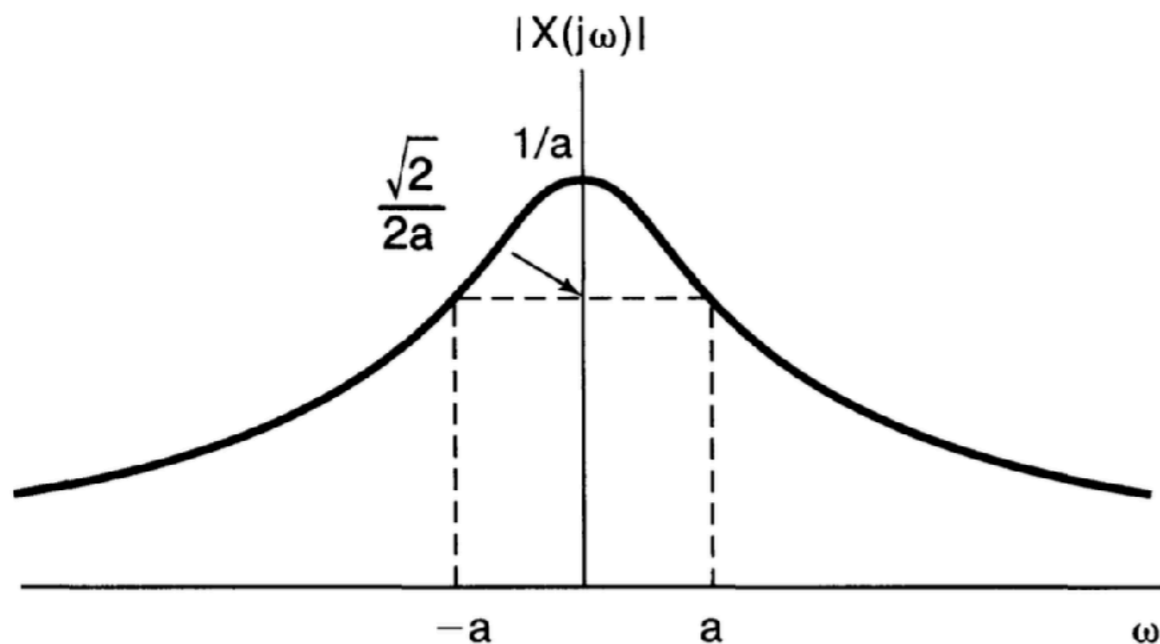
Ejemplos

$$x(t) = e^{-at}u(t) \quad a > 0 \quad X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$x(t) = e^{-at}u(t), \quad a > 0 \quad \xleftrightarrow{\text{TF}} \quad X(j\omega) = \frac{1}{a+j\omega}, \quad a > 0$$

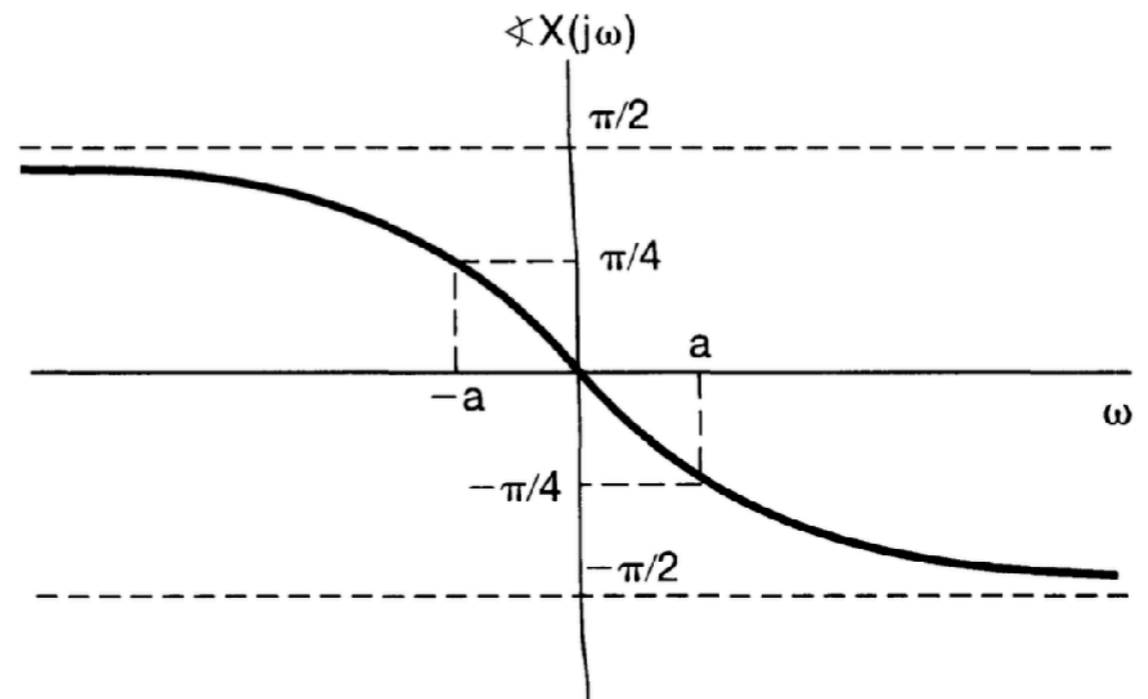
Módulo

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



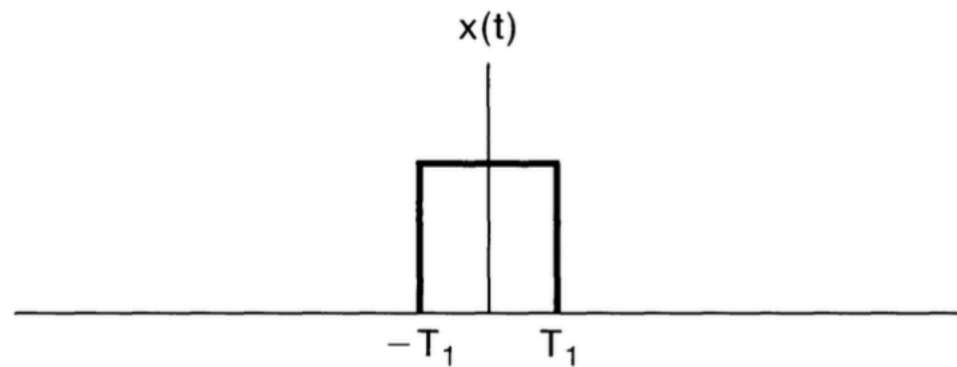
Fase

$$\angle X(j\omega) = -\arctan\left(\frac{\omega}{a}\right)$$

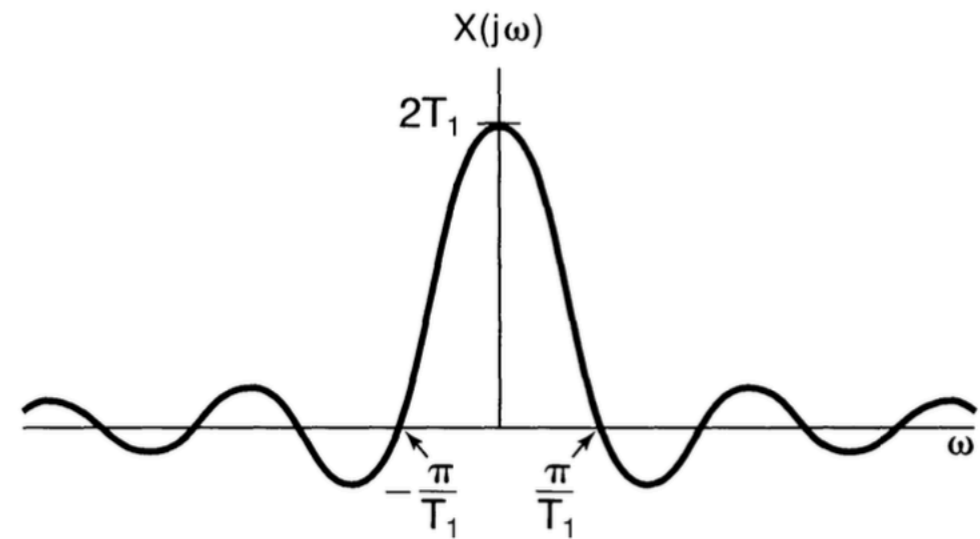


Ejemplos

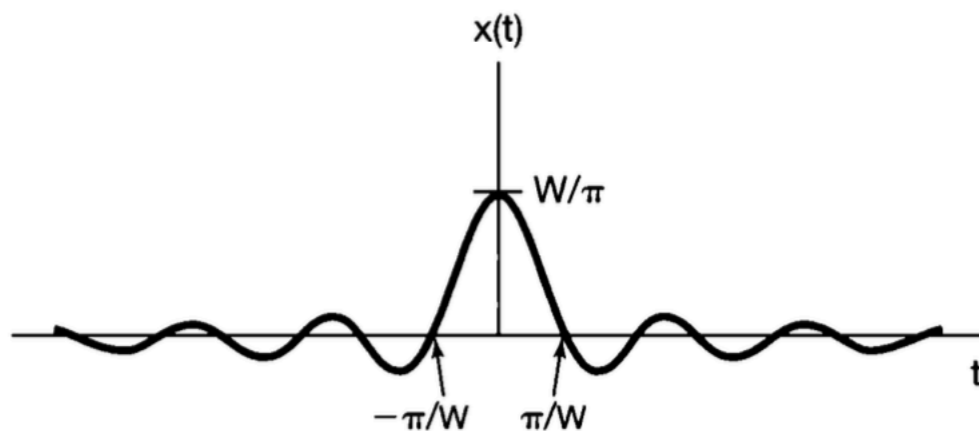
Ejemplo 4.4 $x(t) = \Pi\left(\frac{t}{2T_1}\right) \xleftrightarrow{\text{TF}} X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega}$



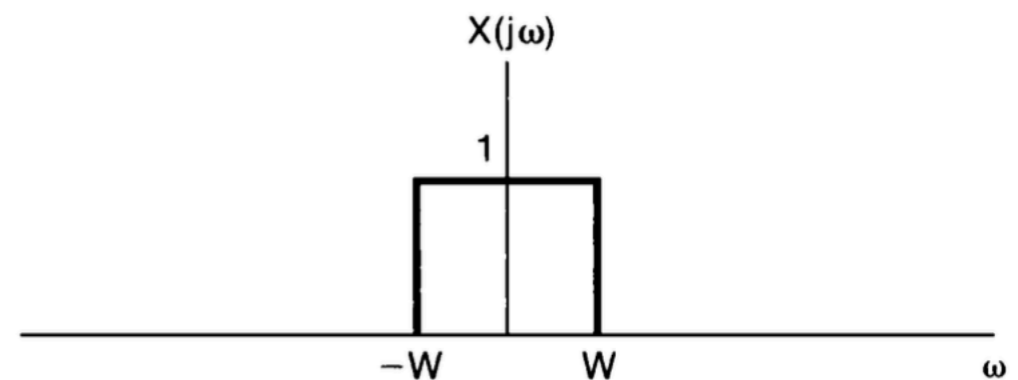
$\xleftrightarrow{\text{TF}}$



Ejemplo 4.5 $X(j\omega) = \Pi\left(\frac{\omega}{2W}\right) \xleftrightarrow{\text{TF}} x(t) = \frac{\sin(Wt)}{\pi t}$

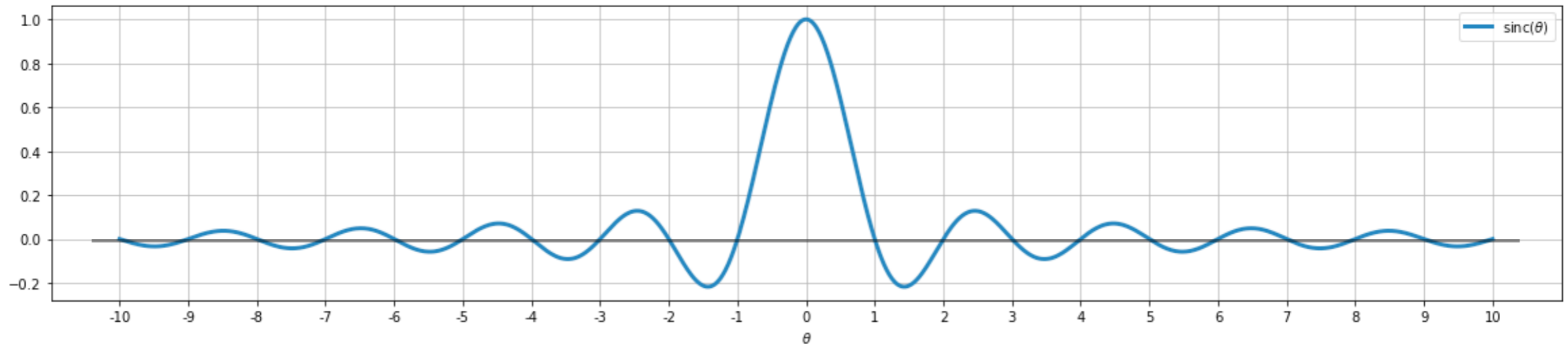


$\xleftrightarrow{\text{TF}}$



Seno cardinal (sinc)

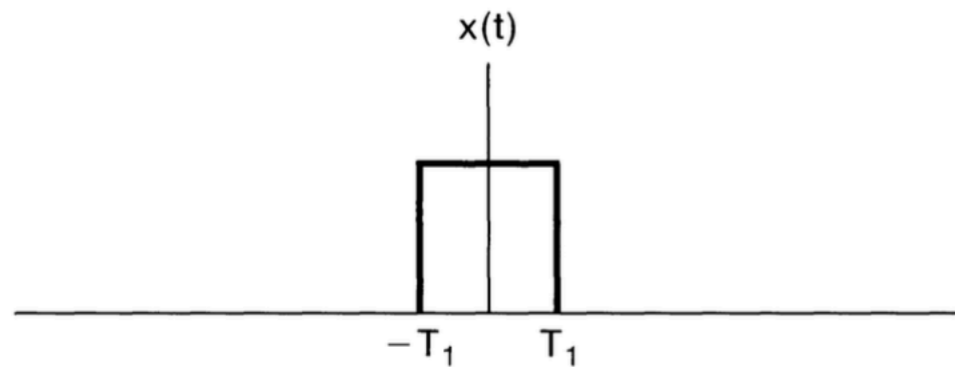
$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$



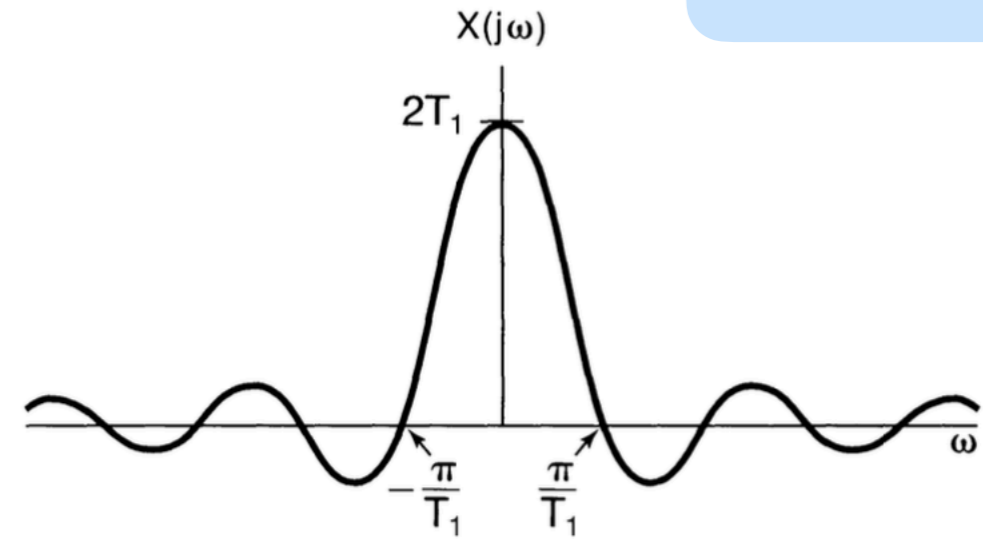
- Área unitaria.
- Vale 1 en $\theta = 0$: $\text{sinc}(0) = 1$
- Cruces periódicos en $\theta \in \mathbb{Z}$.
- Soporte infinito.

Ejemplos

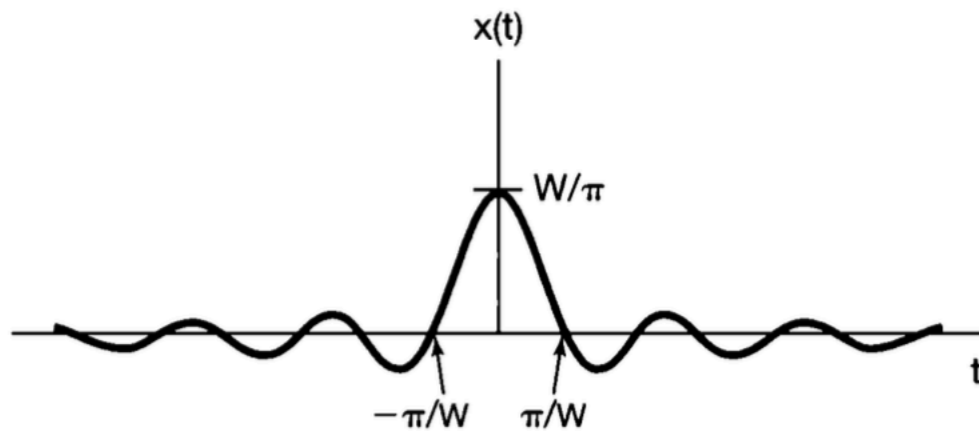
$$x(t) = \Pi\left(\frac{t}{2T_1}\right) \xleftrightarrow{\text{TF}} X(j\omega) = 2\frac{\sin(\omega T_1)}{\omega} = 2T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right)$$



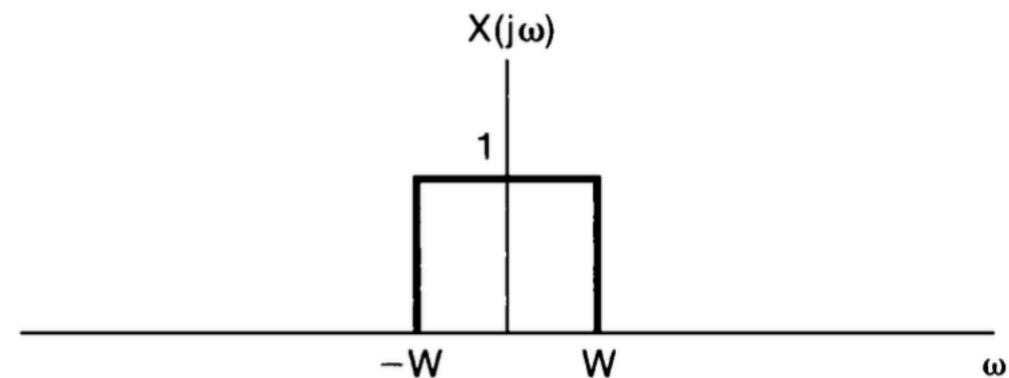
$\xleftrightarrow{\text{TF}}$



$$X(j\omega) = \Pi\left(\frac{\omega}{2W}\right) \xleftrightarrow{\text{TF}} x(t) = \frac{\sin(Wt)}{\pi t} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

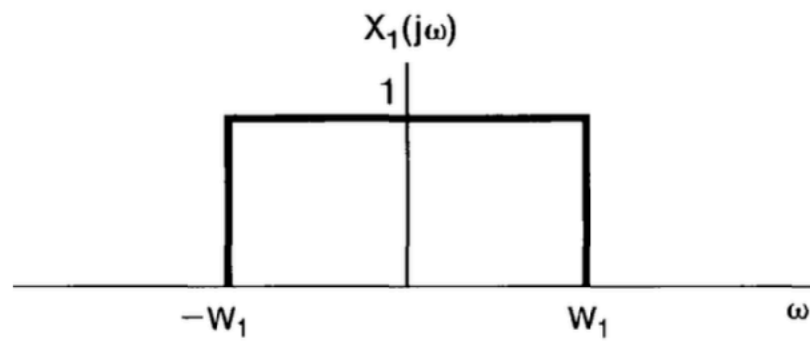
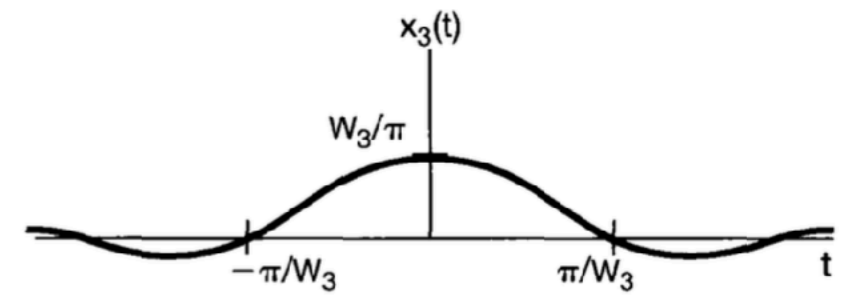
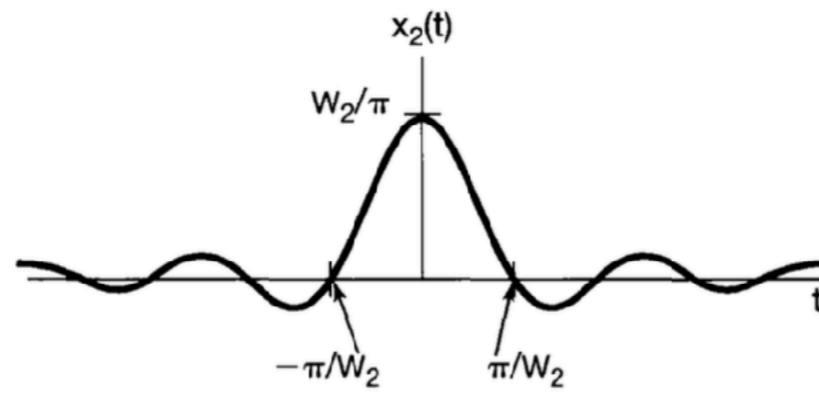
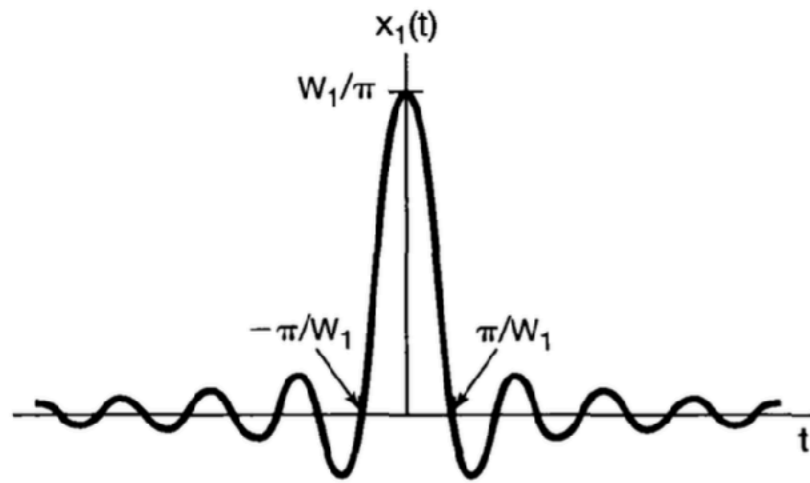


$\xleftrightarrow{\text{TF}}$

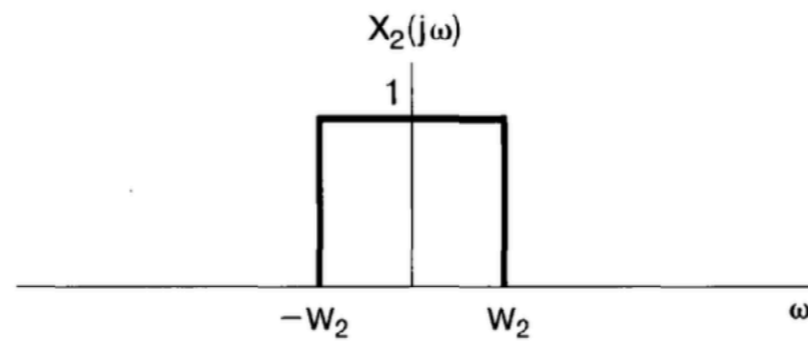


Ejemplos

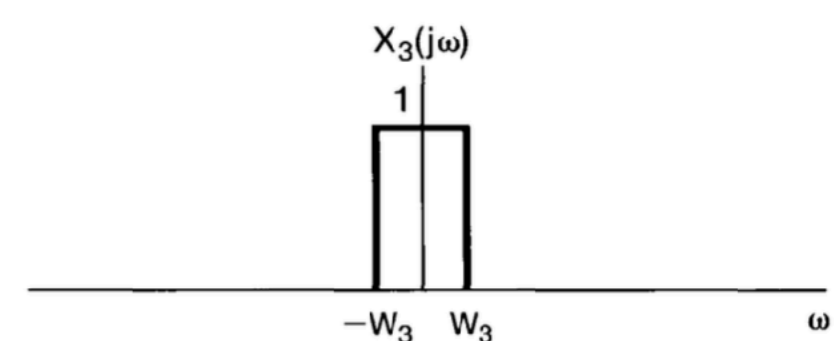
$$W_1 > W_2 > W_3$$



(a)



(b)

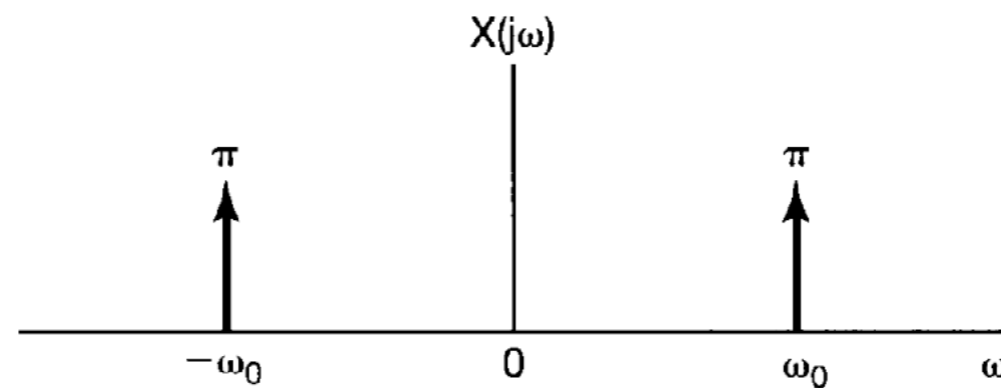


(c)

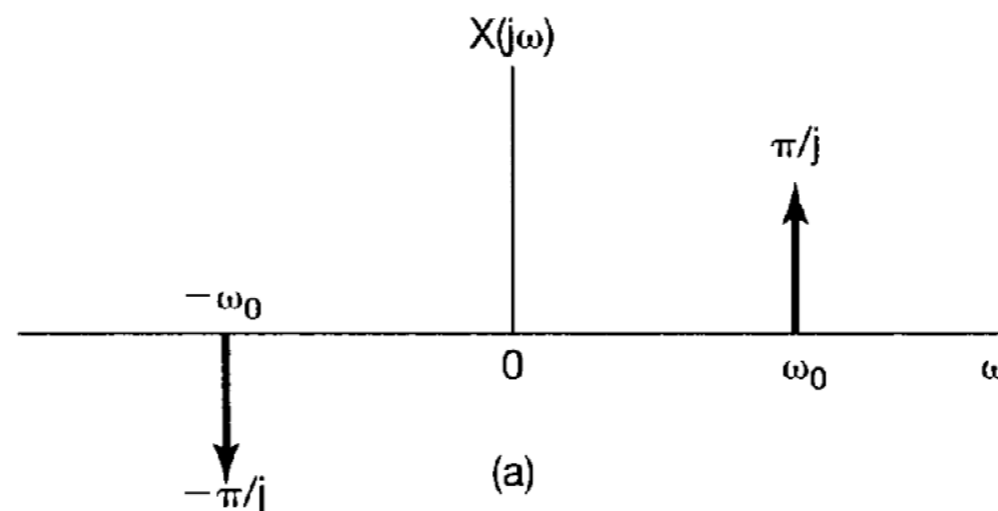
Ejemplos

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \quad \xleftrightarrow{\text{TF}} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{-j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = \cos(\omega_0 t) \quad \xleftrightarrow{\text{TF}} \quad X(j\omega) = \pi (\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$

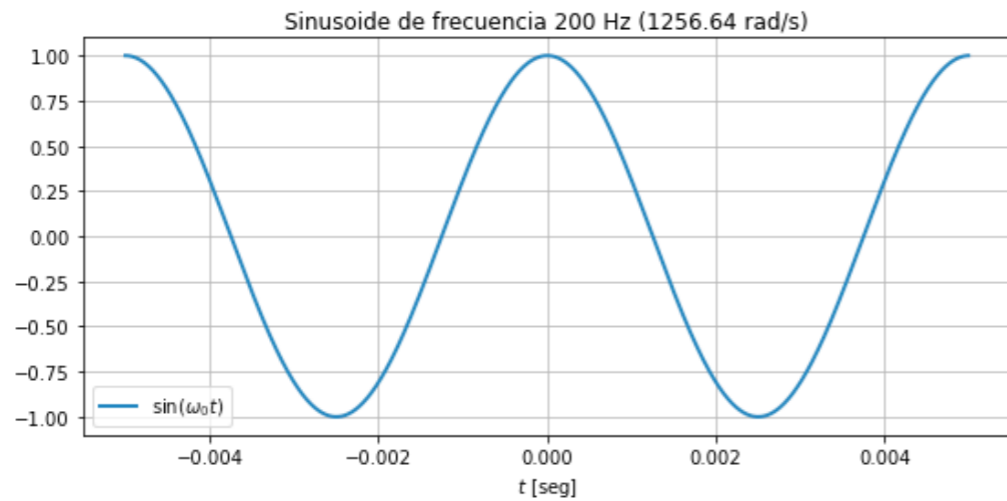


$$x(t) = \sin(\omega_0 t) \quad \xleftrightarrow{\text{TF}} \quad X(j\omega) = \frac{\pi}{j} (-\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$

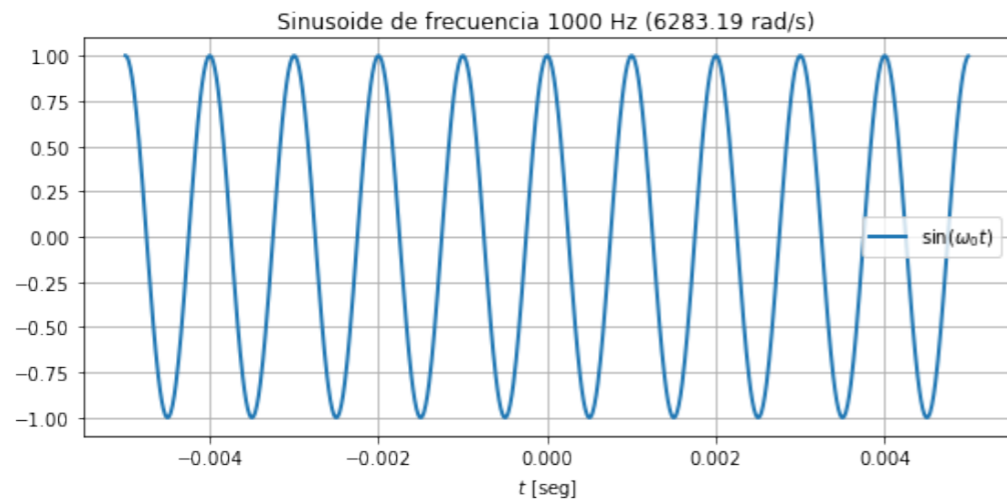
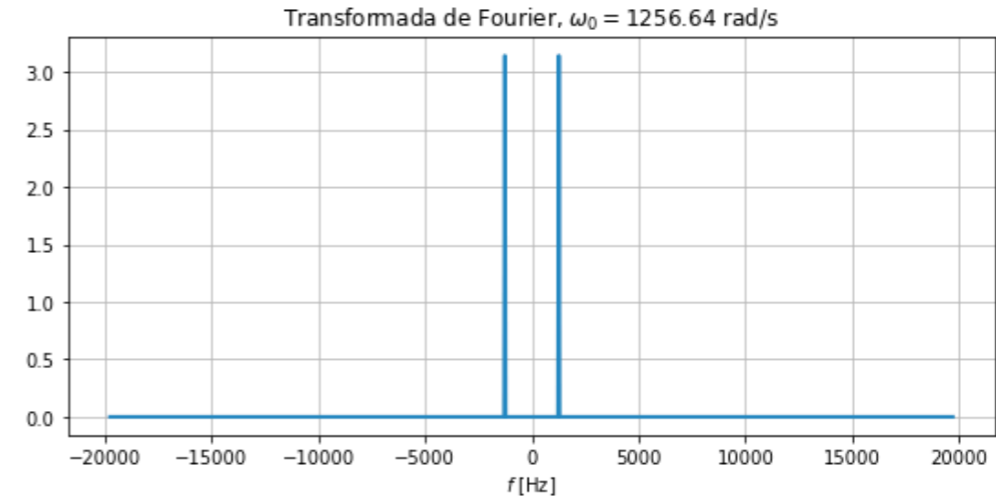


Ejemplos

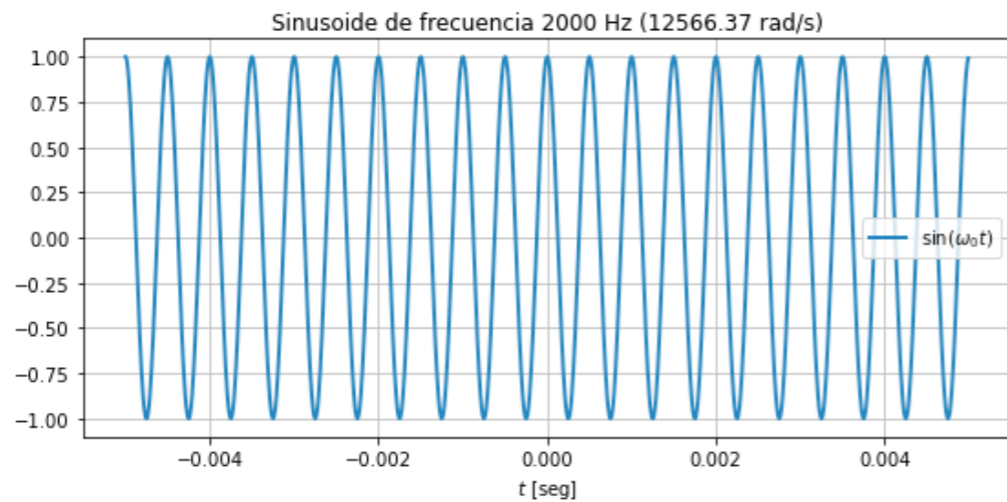
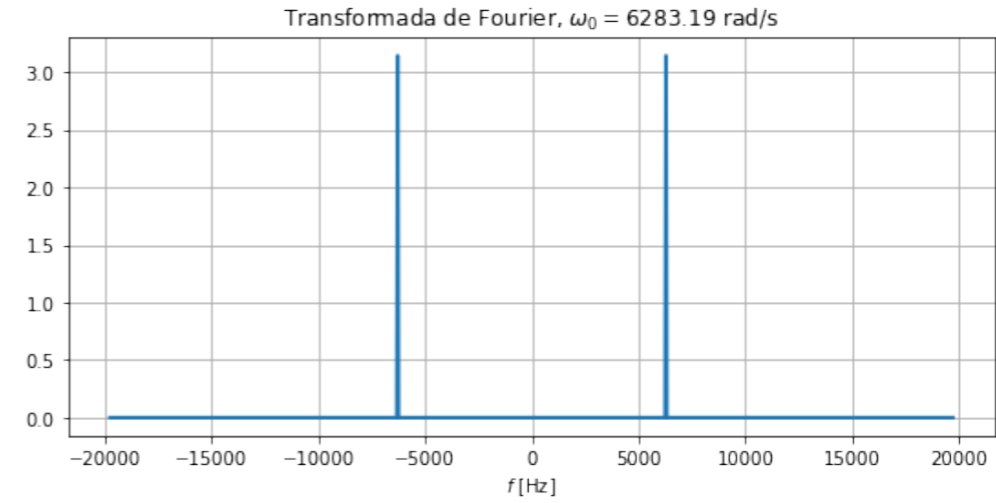
$$x(t) = \cos(\omega_0 t) \quad \xleftrightarrow{\text{TF}} \quad X(j\omega) = 2\pi (-\delta(\omega + \omega_0) + \delta(\omega - \omega_0))$$



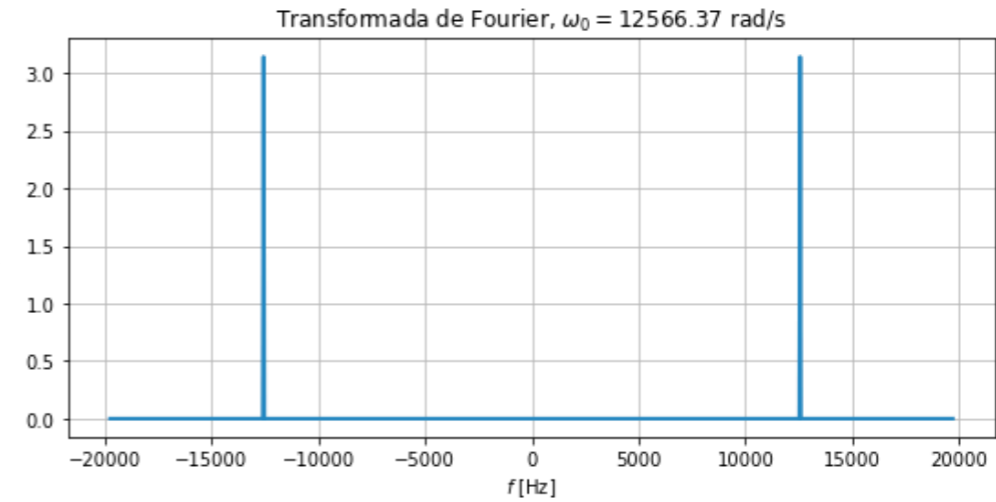
TF



TF



TF



CTFT de una señal periódica

$$X(j\omega) = \sum_k 2\pi a_k \delta(\omega - k\omega_0) \quad \xleftrightarrow{\text{TF}} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - k\omega_0) e^{j\omega t} d\omega$$
$$= \sum_k a_k e^{jk\omega_0 t}$$

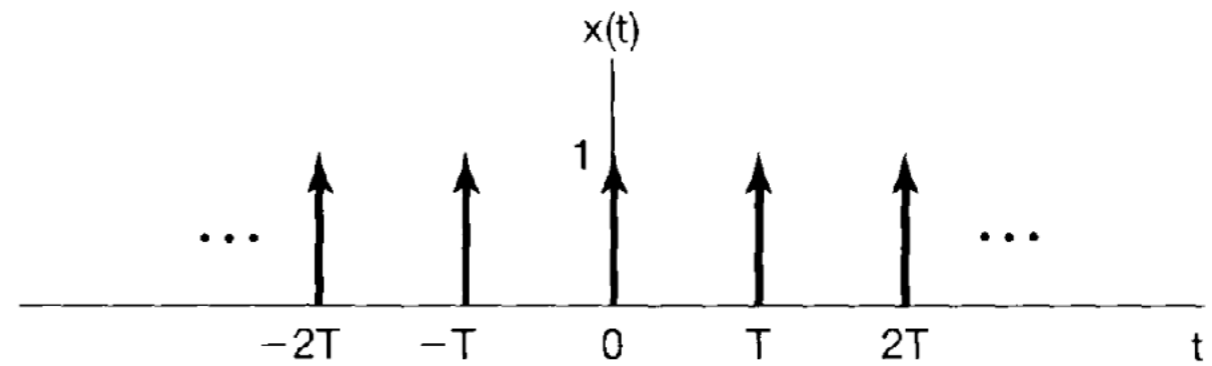
$x(t)$ admite una representación en Serie de Fourier, es una señal periódica de periodo $T_0 = 2\pi/\omega_0$.

La Transformada de Fourier de una **señal periódica** es un **tren de impulsos** en el dominio de la frecuencia, con las áreas de los impulsos proporcionales (2π) a los **coeficientes de la Serie de Fourier**.

Ejemplos

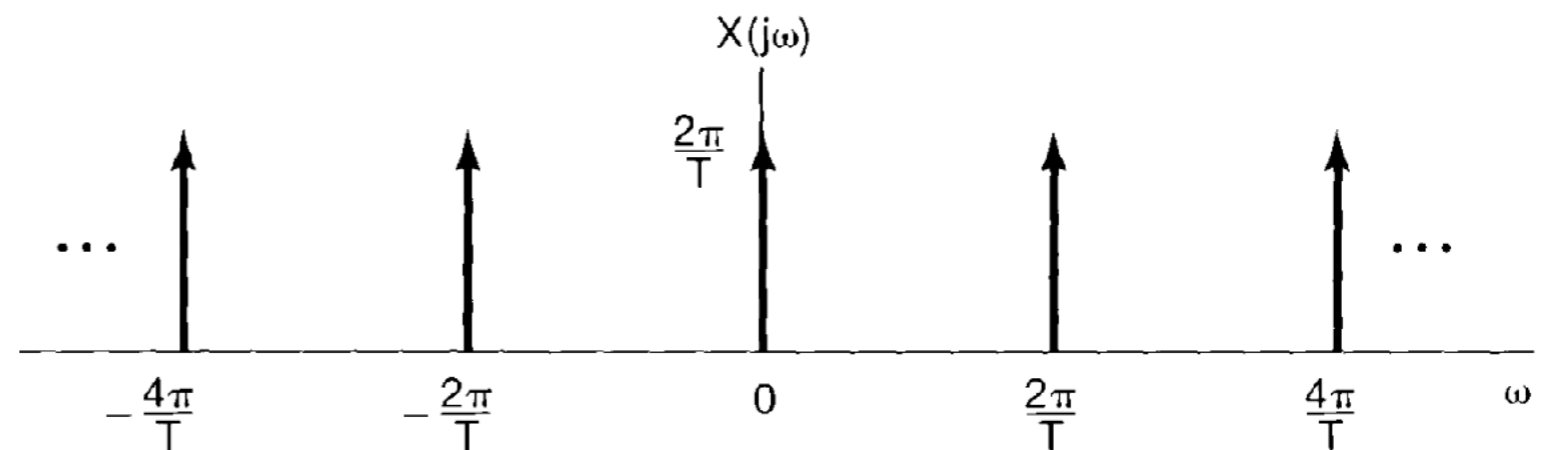
- Peine de Dirac o Tren de Impulsos

$$p_T(t) = \sum_k \delta(t - kT)$$



Serie de Fourier: $p_T[k] = \frac{1}{T} \forall k$

$$P_T(j\omega) = \frac{2\pi}{T} \sum_k \delta(\omega - k\omega_0)$$



CTFT: propiedades

$$x(t) \xleftrightarrow{\text{TF}} X(j\omega) = \mathcal{F}\{x\}$$

$$y(t) \xleftrightarrow{\text{TF}} Y(j\omega) = \mathcal{F}\{y\}$$

- Linealidad

$$a x(t) + b y(t) \xleftrightarrow{\text{TF}} a X(j\omega) + b Y(j\omega)$$

- Desplazamiento temporal

$$x(t - t_0) \xleftrightarrow{\text{TF}} e^{-j\omega t_0} X(j\omega)$$

- Conjugación

$$x^*(t) \xleftrightarrow{\text{TF}} X^*(-j\omega)$$

$x(t)$ es real

$$X(-j\omega) = X^*(j\omega)$$

- Diferenciación

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{TF}} j\omega X(j\omega)$$

- Integración

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\text{TF}} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

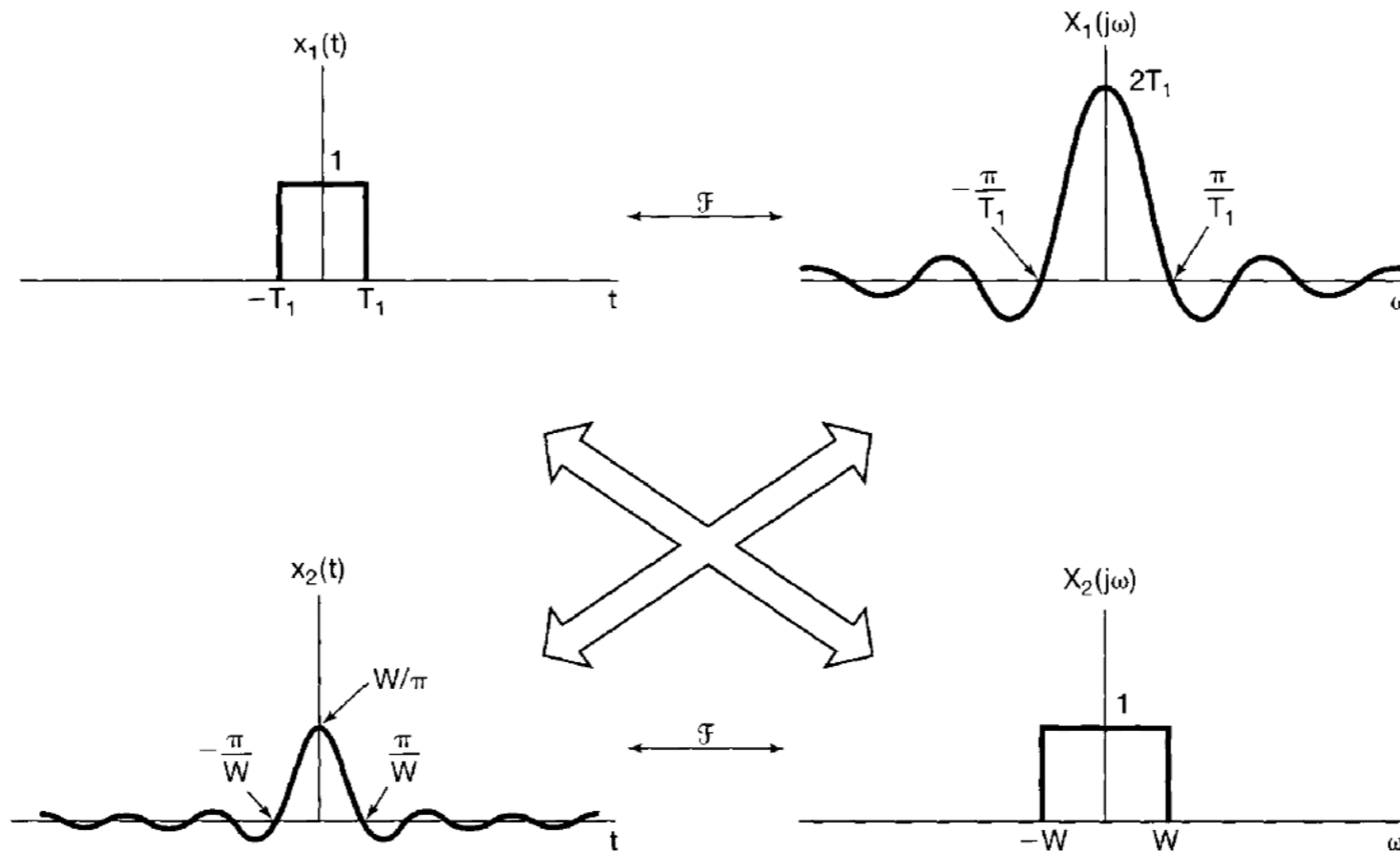
- Escalado tiempo y frecuencia

$$x(at) \xleftrightarrow{\text{TF}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

CTFT: propiedades

$$x(t) \xleftrightarrow{\text{TF}} X(j\omega) = \mathcal{F}\{x\}$$

- Dualidad



$$-jtx(t) \xleftrightarrow{\text{TF}} \frac{dX(j\omega)}{d\omega}$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\text{TF}} X(j(\omega - \omega_0))$$

$$-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \xleftrightarrow{\text{TF}} \int_{-\infty}^{\omega} x(\eta) d\eta$$

CTFT: propiedades

$$x(t) \xleftrightarrow{\text{TF}} X(j\omega) = \mathcal{F}\{x\}$$

$$y(t) \xleftrightarrow{\text{TF}} Y(j\omega) = \mathcal{F}\{y\}$$

$$h(t) \xleftrightarrow{\text{TF}} H(j\omega) = \mathcal{F}\{h\}$$

- Identidad de Parseval

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

- **Convolución**

$$y(t) = h(t) * x(t) \xleftrightarrow{\text{TF}} Y(j\omega) = H(j\omega)X(j\omega)$$

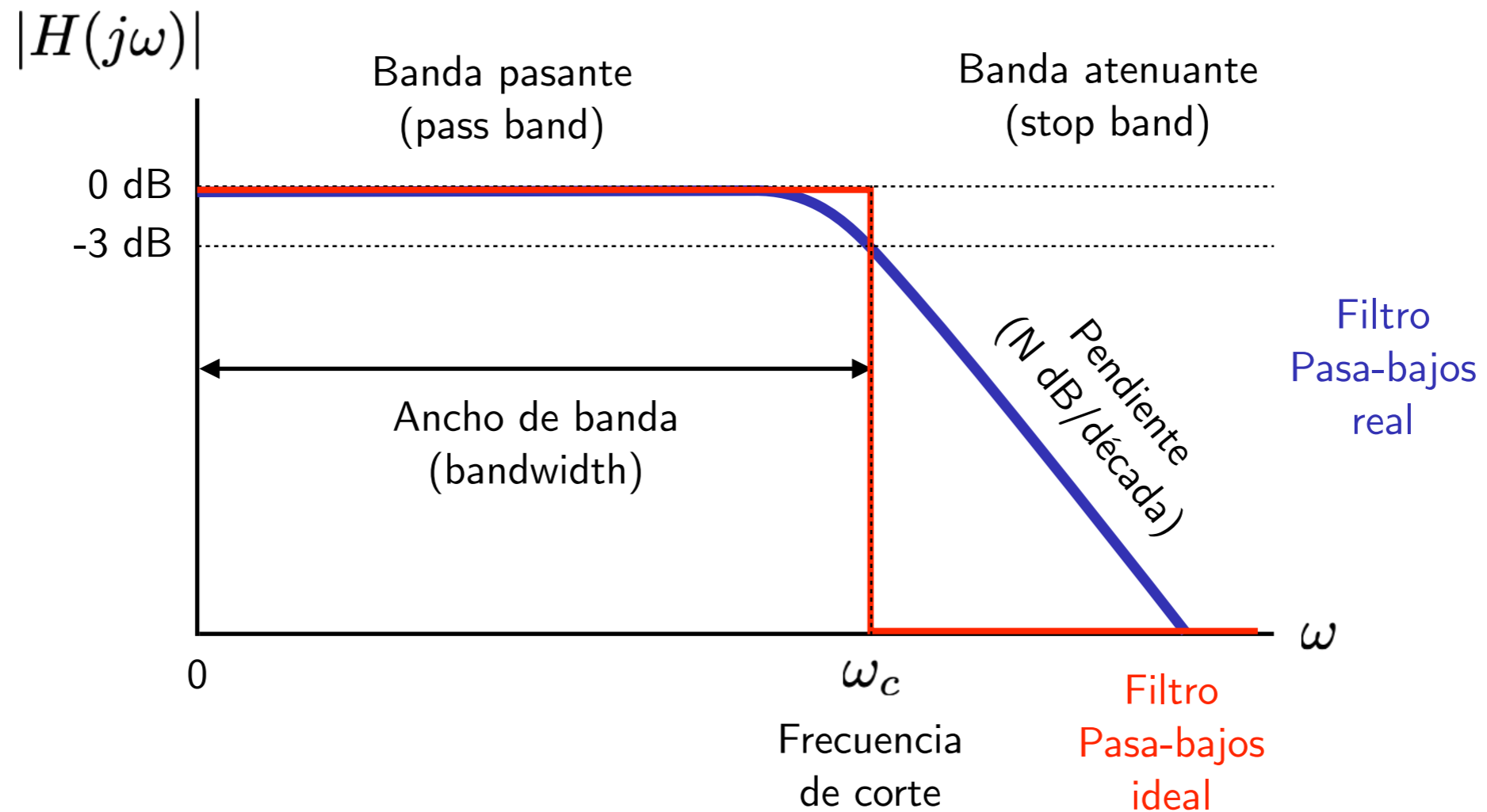
- $H(j\omega)$ es la **respuesta frecuencial del sistema**, existe si $h(t)$ es absolutamente integrable (para sistemas físicos)
 - un SLIT BIBO estable va a tener **respuesta frecuencial $H(j\omega)$**
- $h(t)$ y $H(j\omega)$ caracterizan el sistema,
- es usual llamar a este producto por $H(j\omega)$ **el filtrado** de la señal $X(j\omega)$.

Filtrado

Decibel es la décima parte de un Bel;
es una medida relativa a un valor P_0

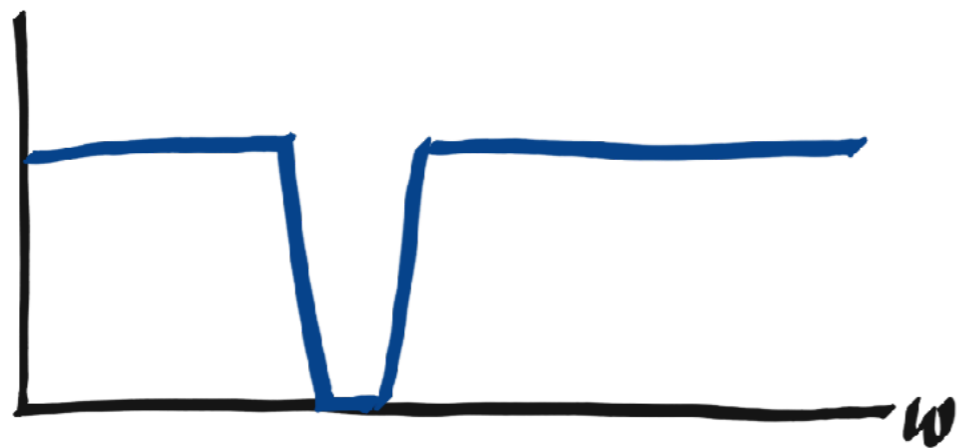
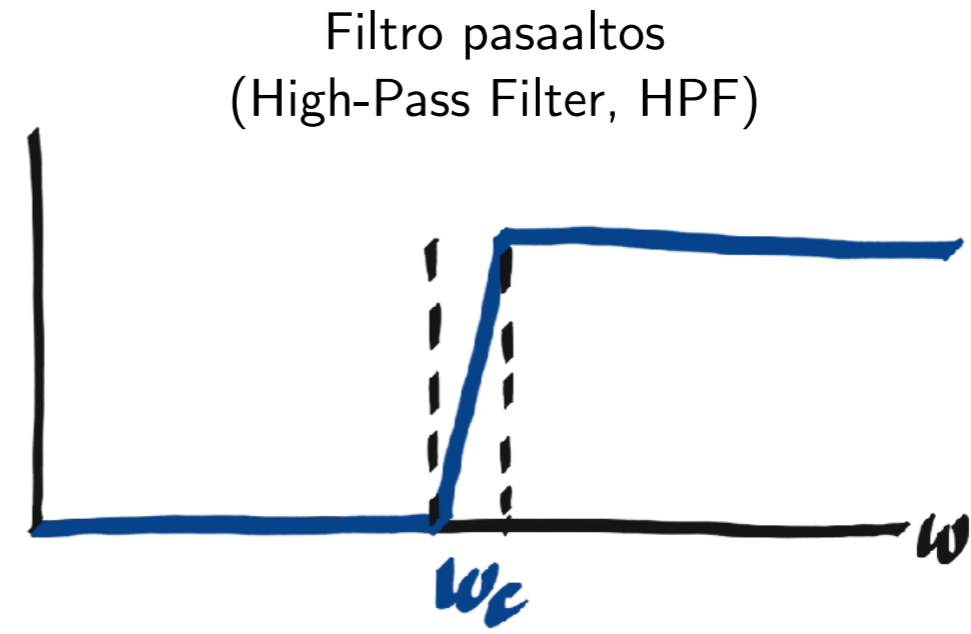
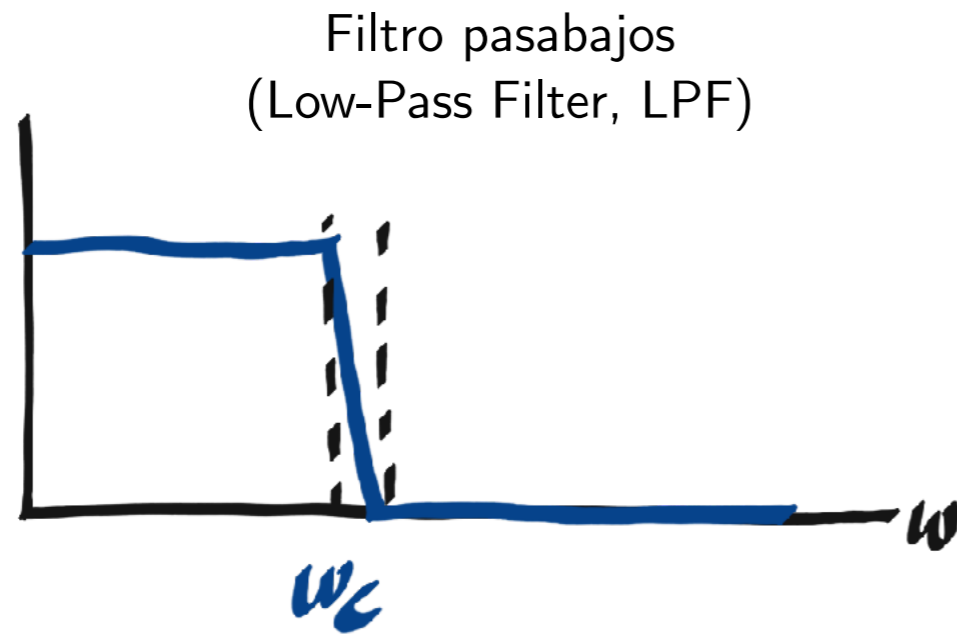
$$\frac{P}{P_0} \text{dB} = 10 \log_{10} \frac{P}{P_0}$$

- Respuesta frecuencia de un filtro $H(j\omega)$:
 - Módulo

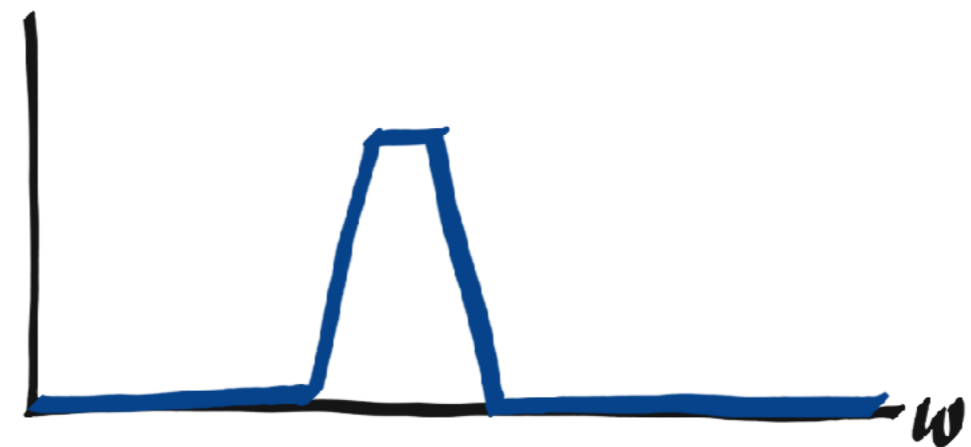


- Igual con la fase $\angle H(j\omega)$

Filtrado

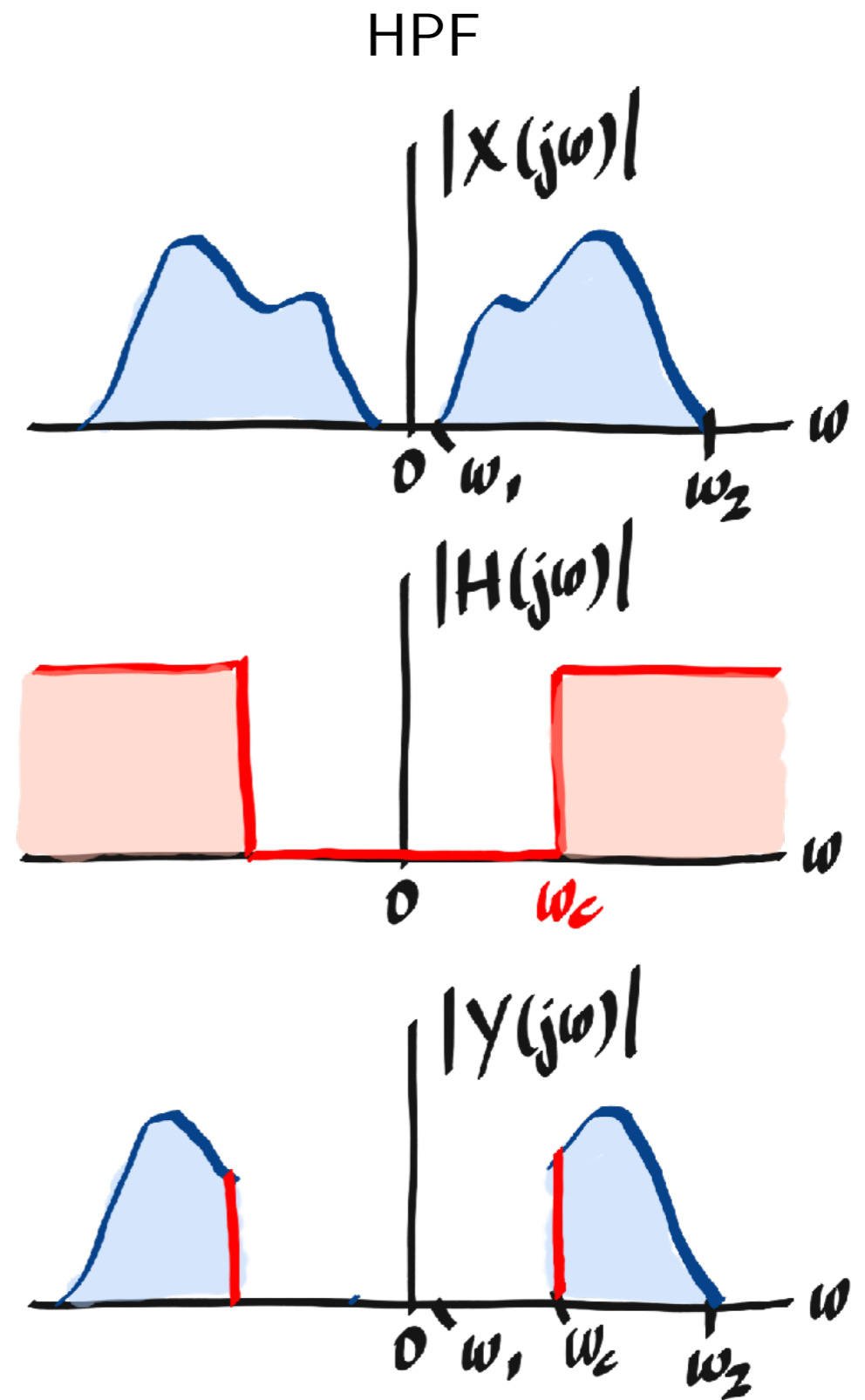
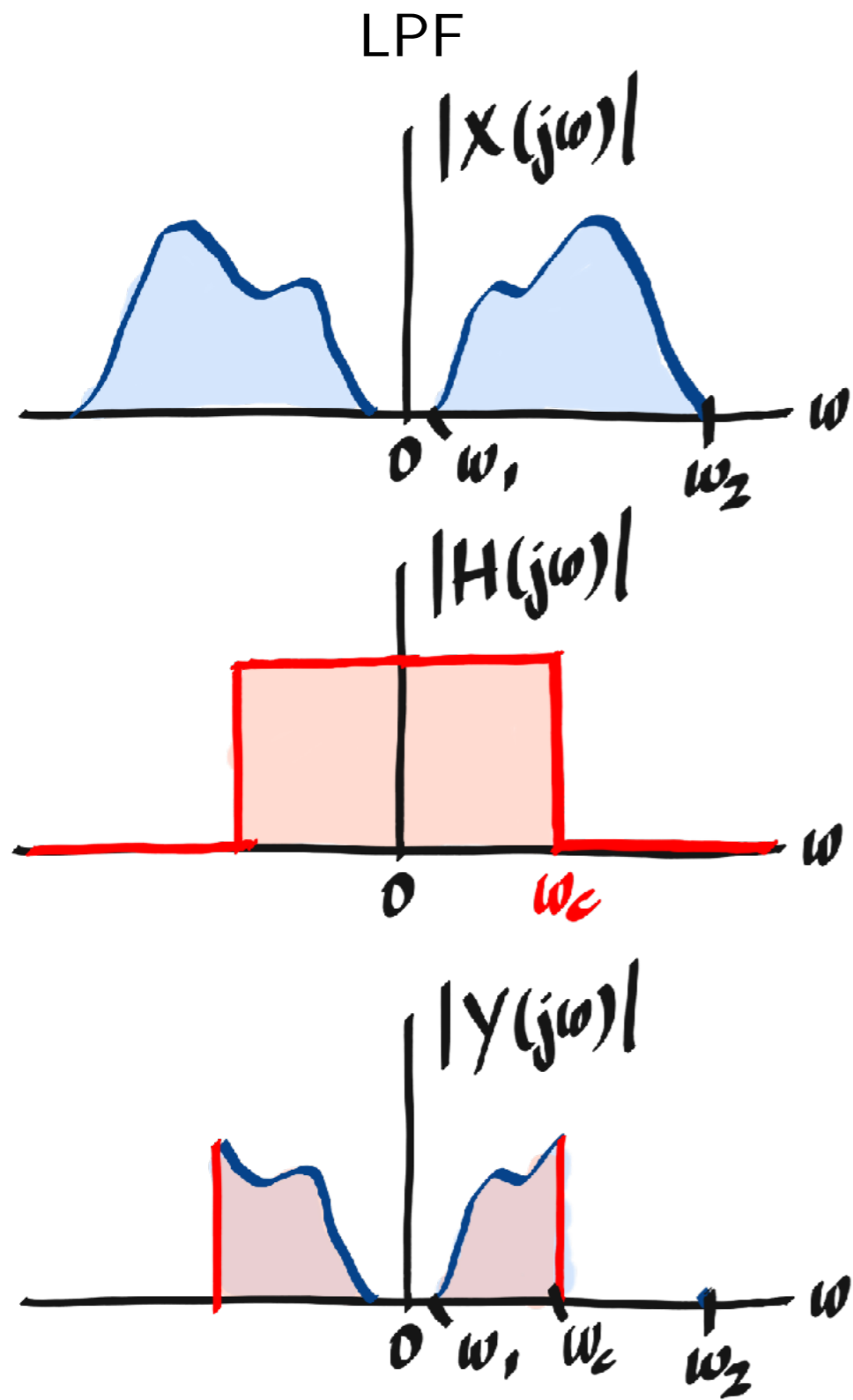


Filtro "matabanda" 😊,
filtro rechaza banda
filtro trampa, filtro notch
(Band-Stop Filter, BSF)



Filtro pasabanda
(Band-Pass Filter, BPF)

Filtrado



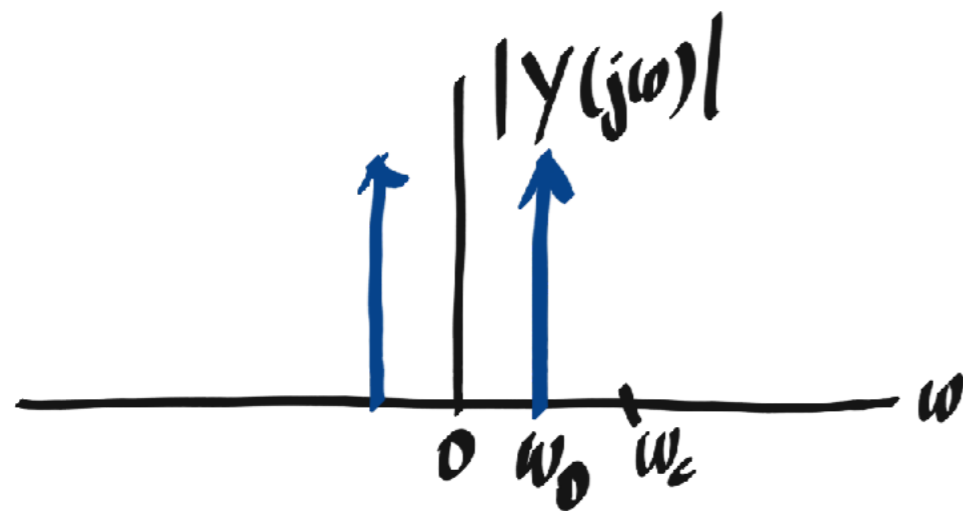
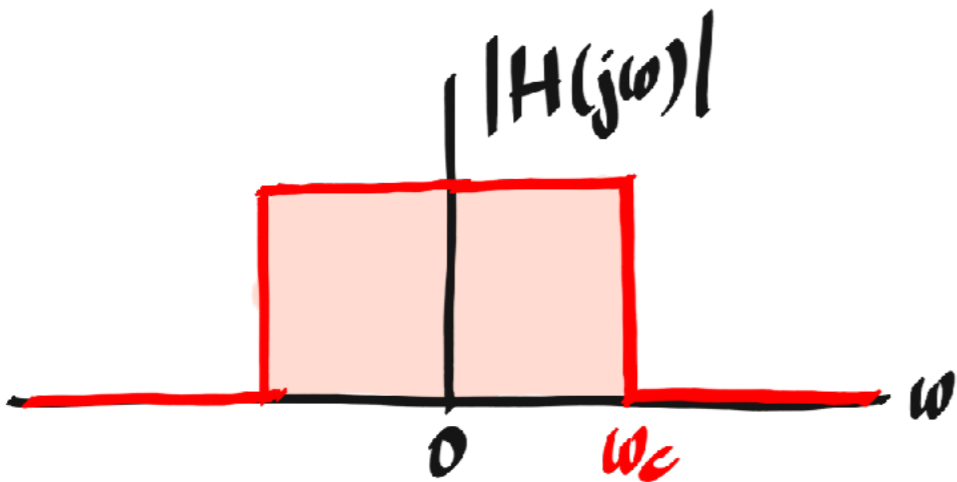
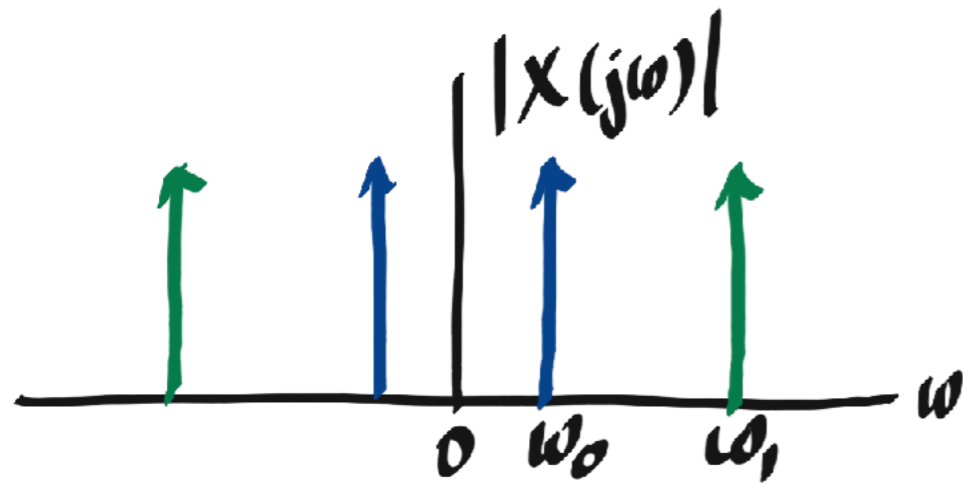
Filtrado

$$x(t) = \cos(\omega_0 t) + \cos(\omega_1 t)$$

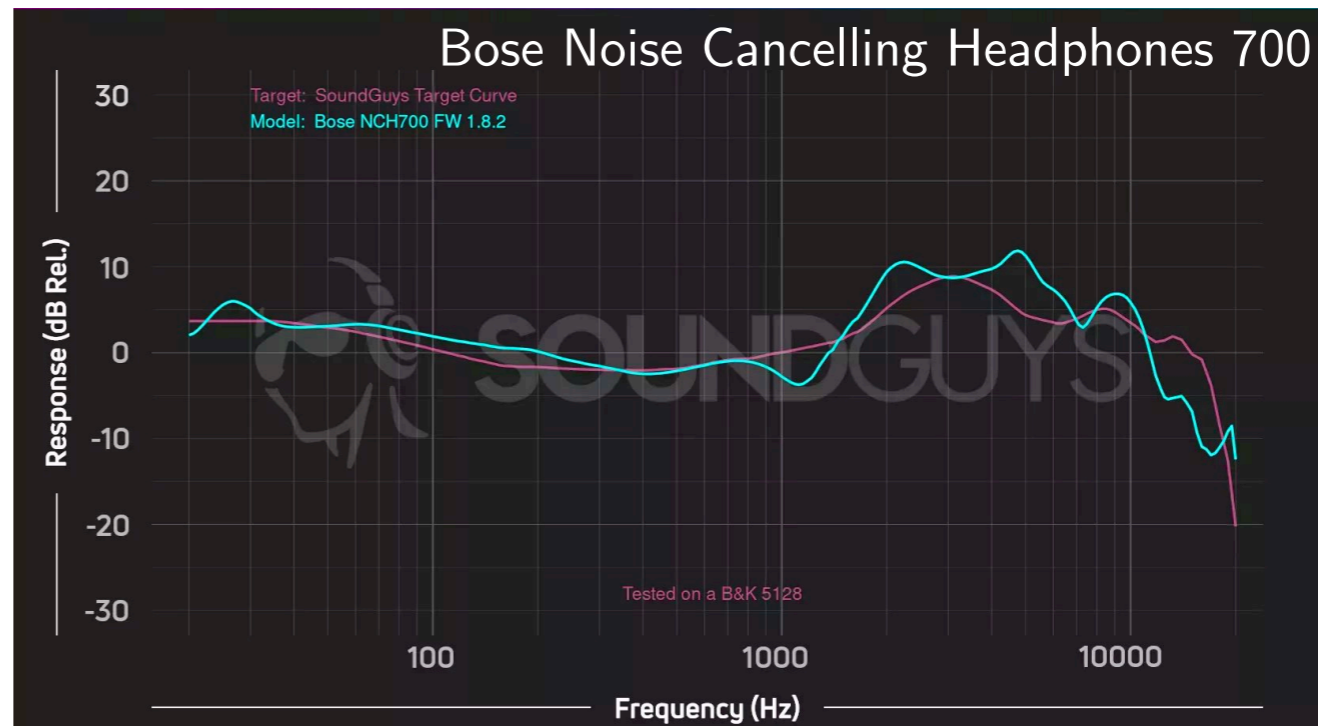
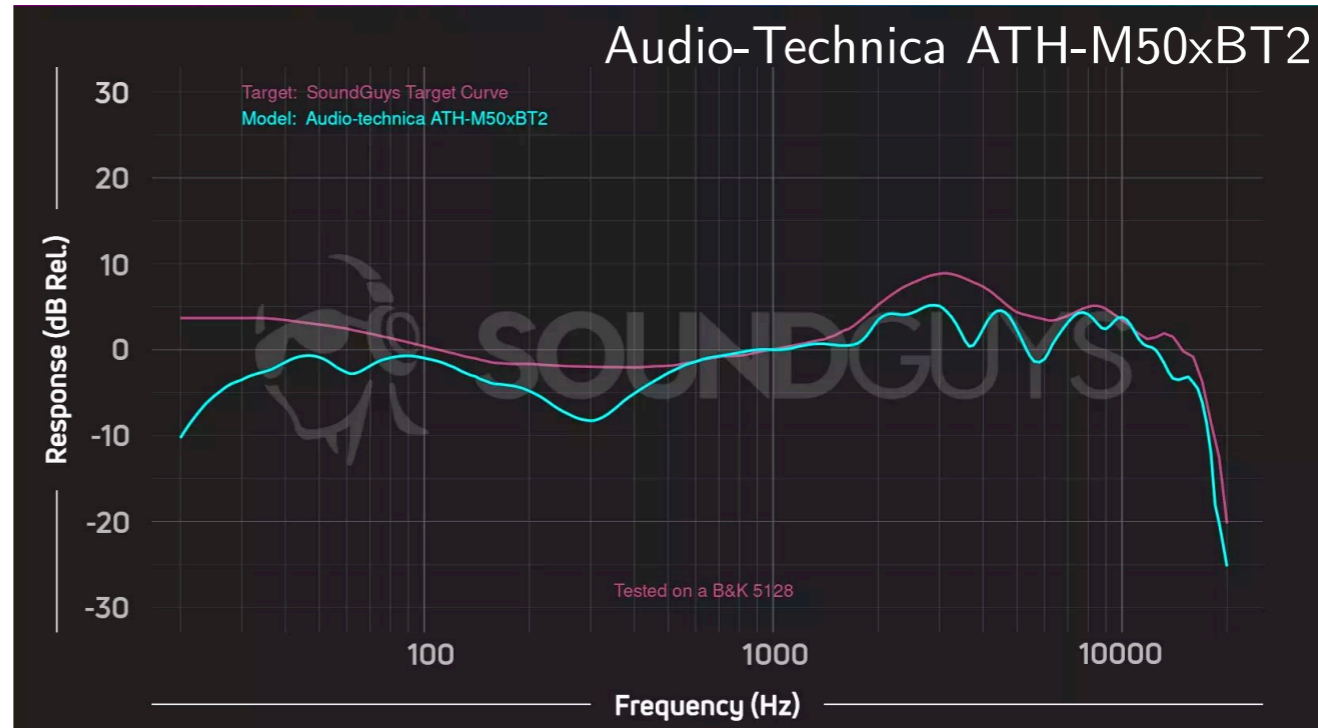
$$h(t) = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \quad \omega_0 < \omega_c < \omega_1$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \left(\cos(\omega_0 t) + \cos(\omega_1 t) \right) * \left(\frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right) \right) \\ &= \cos(\omega_0 t) \end{aligned}$$

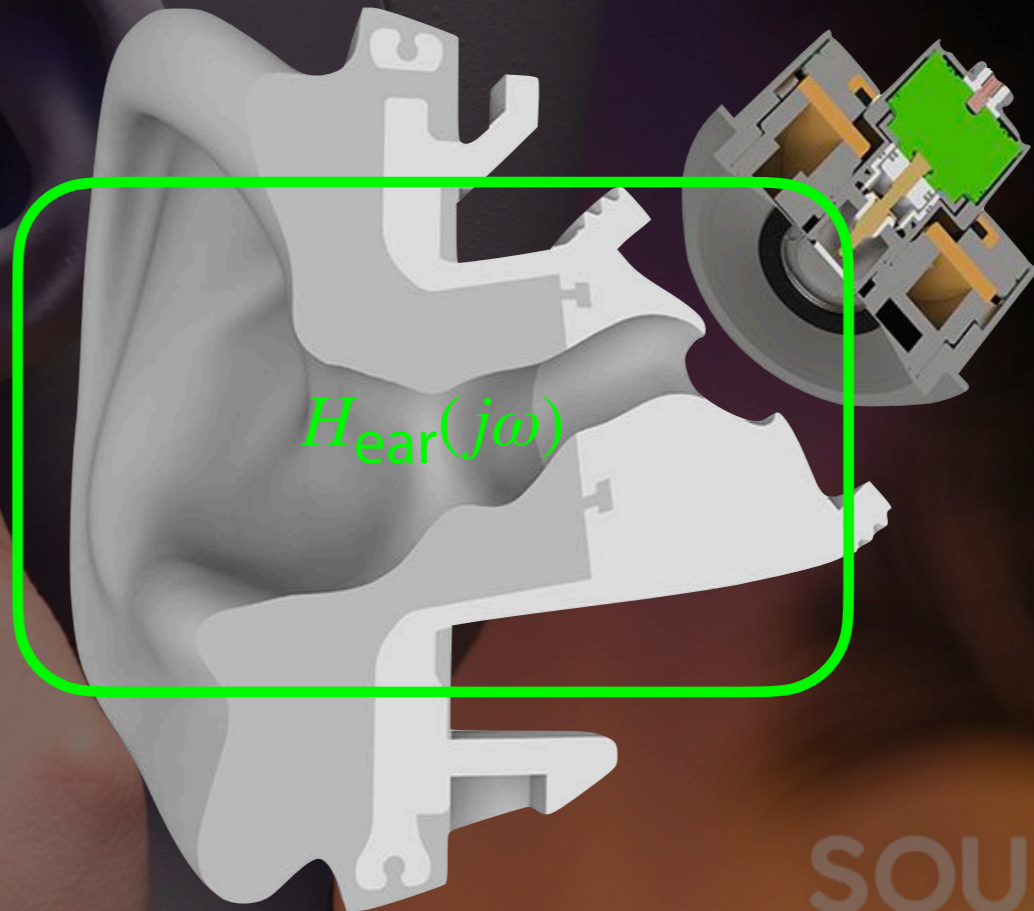
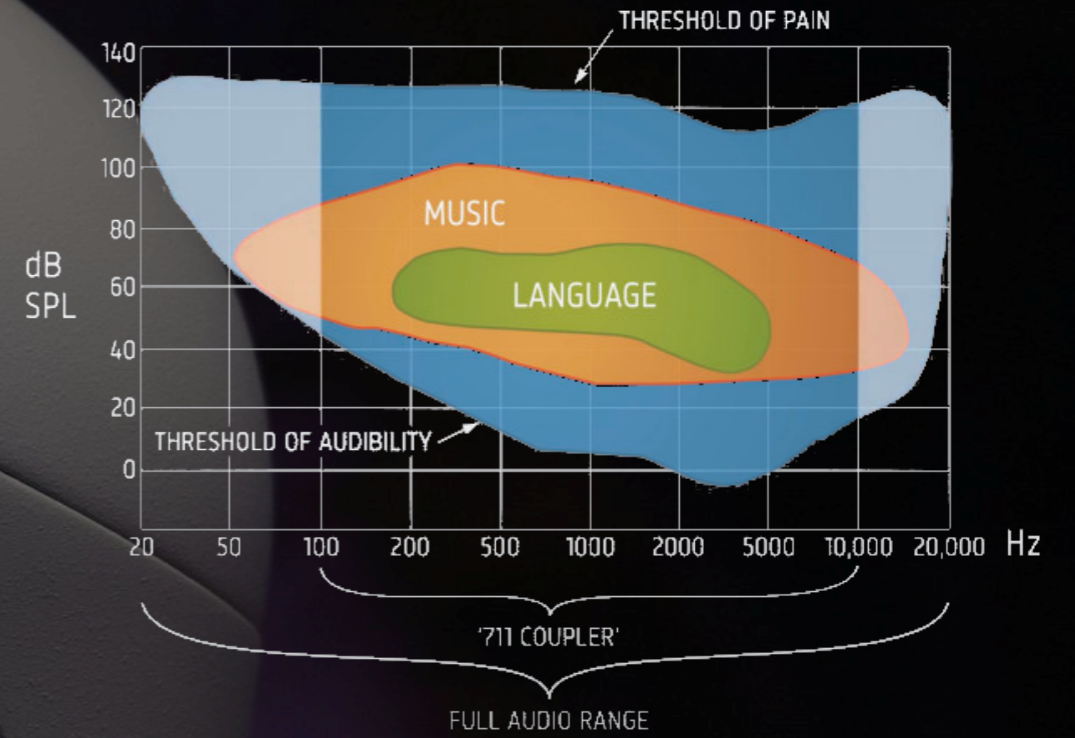
$$Y(j\omega) = X(j\omega) * H(j\omega)$$



Respuesta frecuencial



Respuesta frecuencial



Respuesta frecuencial



Equalizer



Electronic

Flat

Hip-Hop

Jazz

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Judy Is a Punk - 2001 Remaster

Ramones

1:18 -0:14



CTFT: propiedades

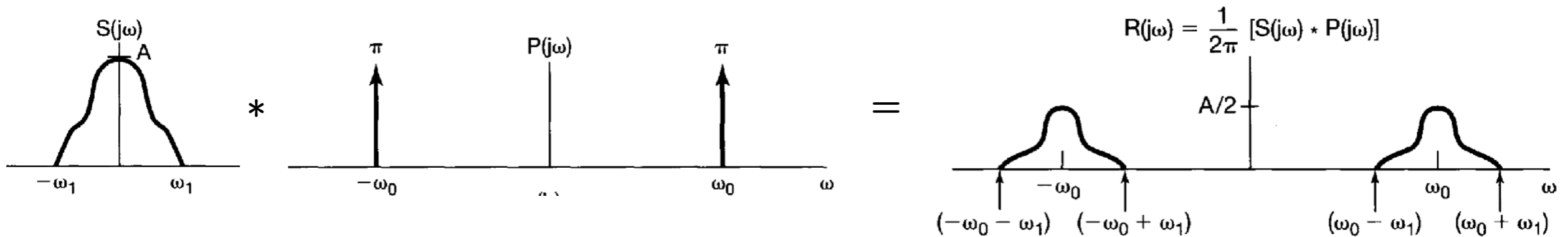
- Multiplicación

$$r(t) = s(t)p(t) \xleftrightarrow{\text{TF}} R(j\omega) = \frac{1}{2\pi} S(j\omega) * P(j\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(j\eta) P(j(\omega - \eta)) d\eta$$

- Modulación

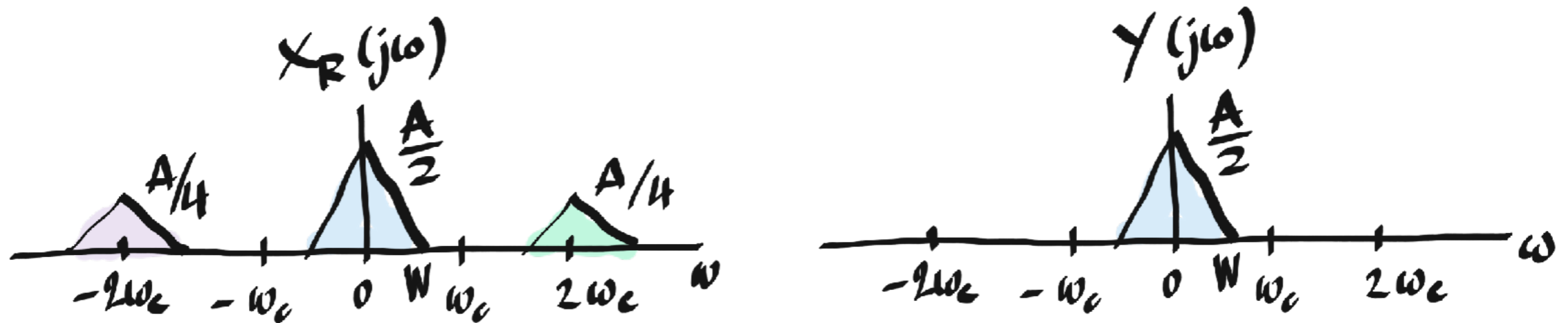
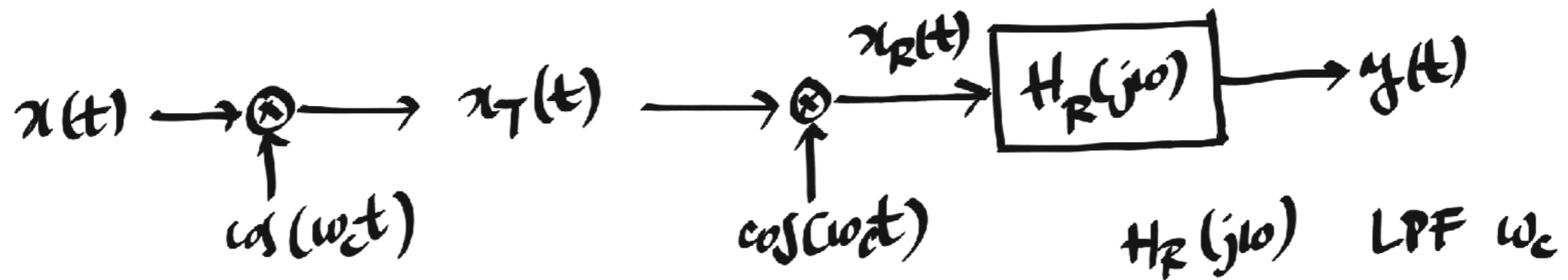
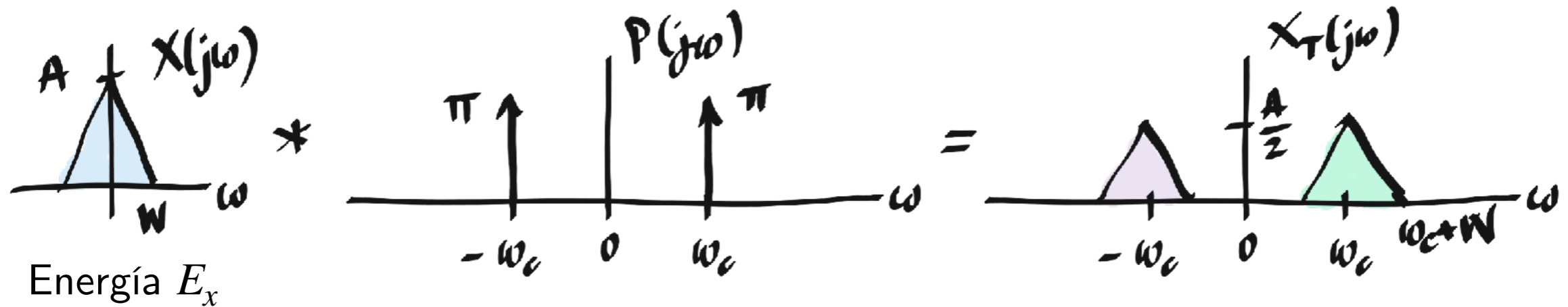
$$p(t) = \cos(\omega_0 t) \xleftrightarrow{\text{TF}} P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



$$x_T(t) = x(t) \cos(\omega_c t) \xleftrightarrow{\text{TF}} X_T(j\omega) = \frac{1}{2} (X(j(\omega + \omega_c)) + X(j(\omega - \omega_c)))$$

Modulación

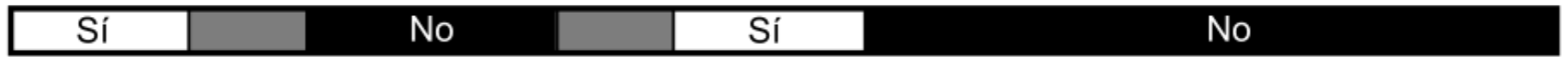
$$x_T(t) = x(t) \cos(\omega_c t) \xleftrightarrow{\text{TF}} X_T(j\omega) = \frac{1}{2} (X(j(\omega + \omega_c)) + X(j(\omega - \omega_c)))$$



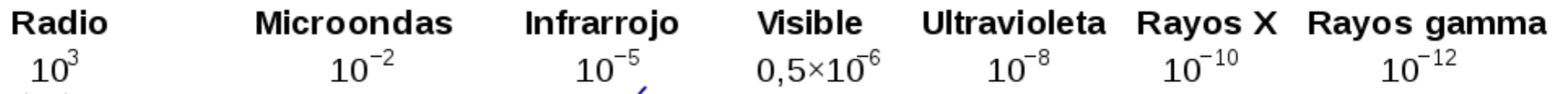
¿Energía de $y(t)$?

Espectro electromagnético

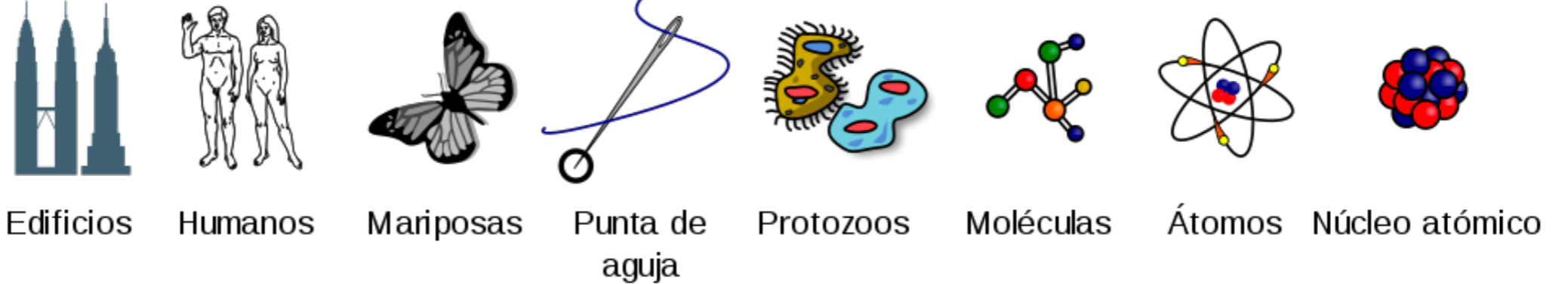
¿Penetra la atmósfera terrestre?



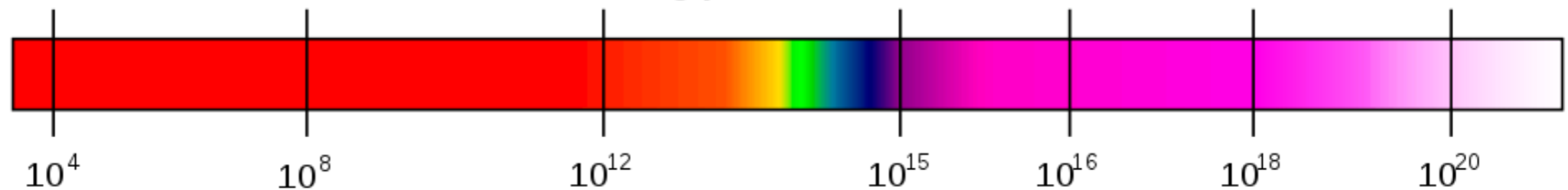
Tipo de radiación
Longitud de onda (m)



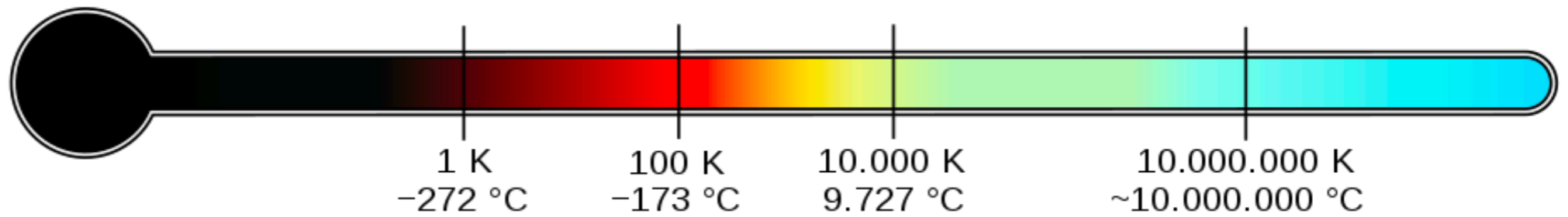
Escala aproximada de la longitud de onda



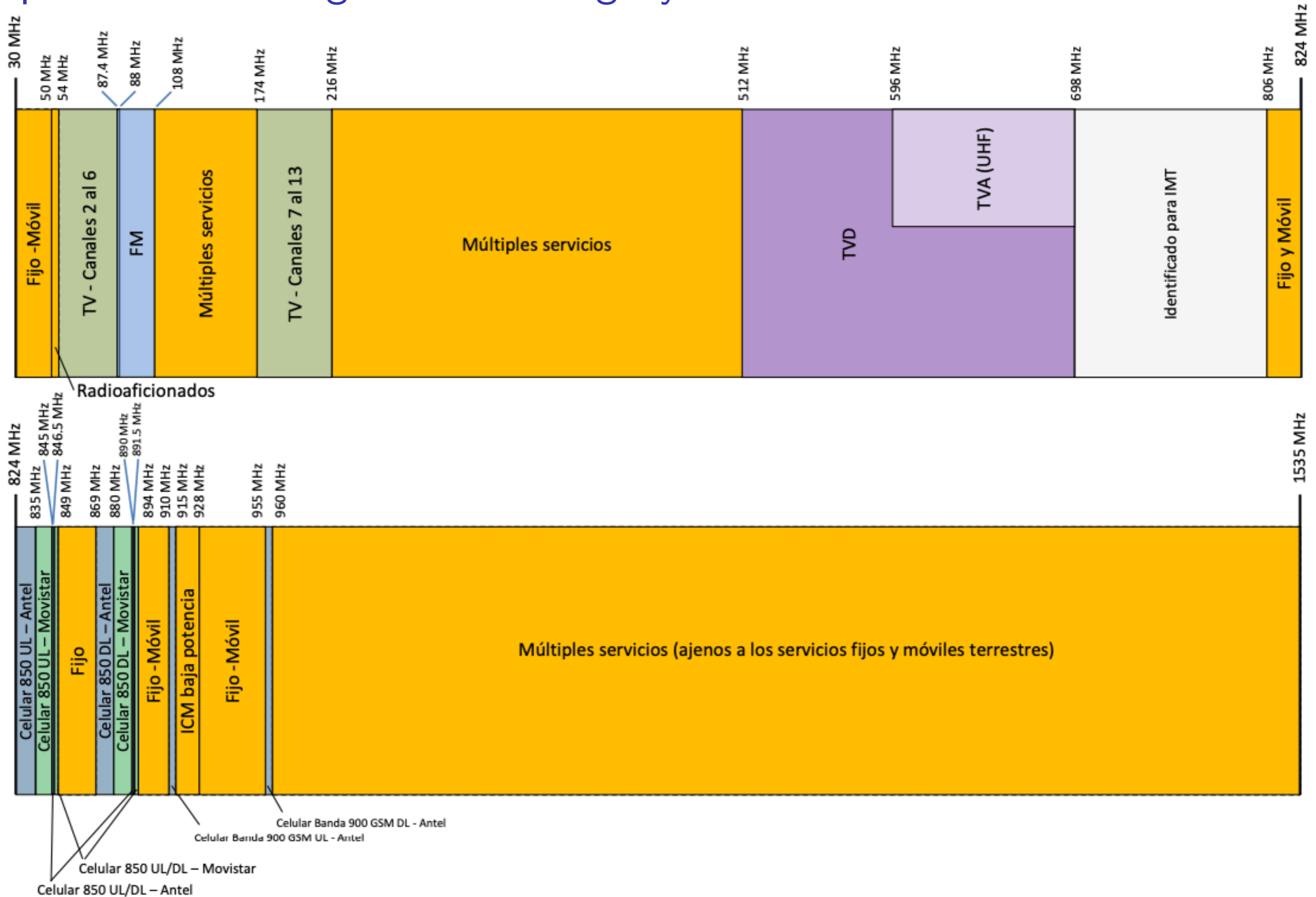
Frecuencia (Hz)



Temperatura de los objetos en los cuales la radiación con esta longitud de onda es la más intensa



Espectro electromagnético en Uruguay



Espectro electromagnético en Uruguay

