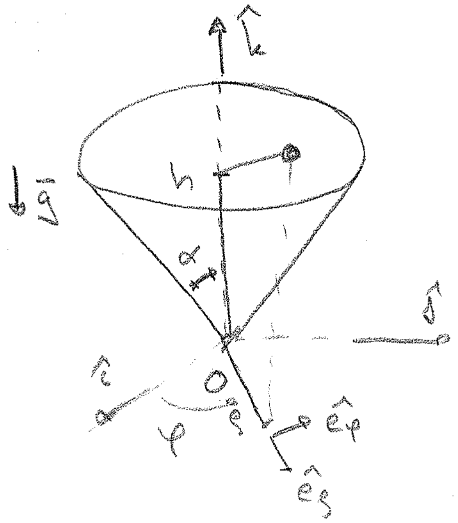
Problemas de clase: Práctico 4

$\boxed{1}$



En coordenadas cilíndricas

$$\bar r = \rho\,\hat e_\rho + h\,\hat k$$

$$\bar v = \dot\rho\,\hat e_\rho + \dot h\,\hat k + \rho\,\dot{\hat e}_\rho$$

$$\bar v = \dot\rho\,\hat e_\rho + \rho\dot\varphi\,\hat e_\varphi + \dot h\,\hat k$$

$$\bar L_O = \bar r \wedge m\bar v = (\rho\,\hat e_\rho + h\,\hat k)\wedge m(\dot\rho\,\hat e_\rho + \rho\dot\varphi\,\hat e_\varphi + \dot h\,\hat k)$$

$$\bar L_O = m\left(\rho^2\dot\varphi\,\hat k + \rho\dot h(-\hat e_\varphi) + h\dot\rho\,\hat e_\varphi - \rho h\dot\varphi\,\hat e_\rho\right)$$

$$\bar L_O = -m\rho h\dot\varphi\,\hat e_\rho + m(h\dot\rho - \rho\dot h)\,\hat e_\varphi + m\rho^2\dot\varphi\,\hat k$$

$$\frac{d\bar L_O}{dt} = \bar r \wedge \bar F_N \qquad \bar F_N = -mg\,\hat k + \bar N$$

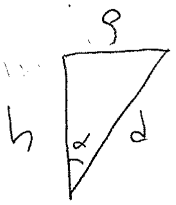
$$\bar N = -N\cos\alpha\,\hat e_\rho + N\sin\alpha\,\hat k$$

$$= (\rho\,\hat e_\rho + h\,\hat k)\wedge\left[(N\sin\alpha - mg)\,\hat k - N\cos\alpha\,\hat e_\rho\right]$$

$$= [N\sin\alpha - mg]\,\rho\,\underbrace{(\hat e_\rho\wedge\hat k)}_{-\hat e_\varphi} - Nh\cos\alpha\,\hat e_\varphi$$

$$= \left[mg - N(\rho\sin\alpha + h\cos\alpha)\right]\hat e_\varphi$$

Vínculo: La partícula se mueve en un cono.



$$d^2 = \rho^2 + h^2 \quad \text{y} \quad tg\,\alpha = \frac{\rho}{h} \rightarrow h\,tg\,\alpha = \rho$$

$$\rightarrow \dot\rho = \dot h\,tg\,\alpha$$

Escribo en función de ρ

$$\bar L_O = -m\rho\frac{\rho}{tg\,\alpha}\dot\varphi\,\hat e_\rho + m\left(\frac{\rho}{tg\,\alpha}\dot\rho - \rho\frac{\dot\rho}{tg\,\alpha}\right)\hat e_\varphi + m\rho^2\dot\varphi\,\hat k$$

$$\bar L_O = -m\frac{\rho^2}{tg\,\alpha}\dot\varphi\,\hat e_\rho + m\rho^2\dot\varphi\,\hat k$$

$$\frac{d\vec{L}_O}{dt} = -\frac{m}{tg\alpha}\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)\hat{e_\varrho} - m\frac{\varrho^2}{tg\alpha}\,\hat{\dot{e_\varrho}} + m\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)\hat{k}$$

$$= -\frac{m}{tg\alpha}\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)\hat{e_\varrho} - m\frac{\varrho^2\dot{\varphi}}{tg\alpha}\,\hat{e_\varphi} + m\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)\hat{k}$$

$\Rightarrow$ $e_\varrho)$ $\quad 0 = -\frac{m}{tg\alpha}\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)$

$\hat{e_\varphi})$ $\quad mg - N\left(\varrho\sin\alpha + \varrho\frac{\cos\alpha}{tg\alpha}\right) = -m\frac{\varrho^2\dot{\varphi}}{tg\alpha}$

$\hat{k})$ $\quad 0 = m\left(2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi}\right)$

Las ec. según $\hat{e_\varrho}$ y $\hat{k}$ dan lo mismo
la conservación de $\vec{L}_O\cdot\hat{k} \Rightarrow m\varrho^2\dot{\varphi} = \ell$

Otra demostración $\qquad$ 2da Ley de Newton.

$\vec{F}_N = m\vec{a}$ $\qquad \vec{a} = (\ddot{\varrho} - \varrho\dot{\varphi}^2)\hat{e_\varrho} + (\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi})\hat{e_\varphi} + \ddot{h}\hat{k}$

$\qquad \vec{a} = (\ddot{\varrho} - \varrho\dot{\varphi}^2)\hat{e_\varrho} + (\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi})\hat{e_\varphi} + \frac{\ddot{\varrho}}{tg\alpha}\hat{k}$

$-mg\hat{k} - N\cos\alpha\,\hat{e_\varrho} + N\sin\alpha\,\hat{k} = m\vec{a}$

$\hat{e_\varrho})$ $\quad -N\cos\alpha = m(\ddot{\varrho} - \varrho\dot{\varphi}^2)$

$\hat{e_\varphi})$ $\quad 0 = m(\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi}) \rightarrow \frac{d(\varrho^2\dot{\varphi})}{dt} = 2\varrho\dot{\varrho}\dot{\varphi} + \varrho^2\ddot{\varphi} = \varrho\underbrace{(2\dot{\varrho}\dot{\varphi} + \varrho\ddot{\varphi})}_{0}$

$\hat{k})$ $\quad N\sin\alpha - mg = m\frac{\ddot{\varrho}}{tg\alpha}$

b) Condición inicial $h(t=0)=h_0$

$\vec{v}(t=0)=v_0\,\hat{e}_\varphi \rightarrow$ perpendicular a la generatriz

$\vec{L}_0\cdot\hat{k}=cte \rightarrow \quad l=m\rho^2\dot{\varphi} \quad cte.$

$$\vec{v}(t=0)=\dot{\rho}(t=0)\,\hat{e}_\rho+\rho(0)\dot{\varphi}(0)\,\hat{e}_\varphi+\dot{h}(0)\hat{k}=v_0\,\hat{e}_\varphi$$

$\Rightarrow \dot{\rho}(t=0)=0\,,\ \dot{h}(0)=0$ Obs. respeta el vínculo $\dot{\rho}=\dot{h}\,tg\alpha$

$\rho(0)=h(0)\,tg\alpha=h_0\,tg\alpha$

$\Rightarrow h_0 tg\alpha\,\dot{\varphi}(0)=v_0 \rightarrow \boxed{\dot{\varphi}(0)=\dfrac{v_0}{h_0 tg\alpha}}$ $\quad \dot{\varphi}(0)=\dot{\varphi}_0$

tengo $\dot{\varphi}=\dfrac{l}{m}\dfrac{1}{\rho^2}$ con $l=m\rho_0^2\dot{\varphi}_0$ $\quad \rho_0=h_0\,tg\alpha$

$\Rightarrow l=m\,tg^2\alpha\,h_0^2\dfrac{v_0}{h_0 tg\alpha}$

$l=m\,tg\alpha\,h_0 v_0$

El peso es conservativo

$$P_N=\vec{N}\cdot\vec{v}=(-N\cos\alpha\,\hat{e}_\rho+N\sin\alpha\,\hat{k})\cdot(\dot{\rho}\,\hat{e}_\rho+\rho\dot{\varphi}\,\hat{e}_\varphi+\dot{h}\hat{k})$$

$$=-N\dot{\rho}\cos\alpha+N\sin\alpha\,\dot{h} \xrightarrow[\dot{\rho}=\dot{h}tg\alpha]{} N\dot{\rho}\,(\sin\alpha-tg\alpha\cos\alpha)=0$$

$\Rightarrow P_N=0$

Tengo fuerzas conservativas (el Peso) y de potencia nula (la normal) $\Rightarrow$ El sistema es conservativo

$U_g=mgh$

$$T=\frac{1}{2}m\vec{v}\cdot\vec{v}=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2+\dot{h}^2)=\frac{1}{2}m(\dot{h}^2tg^2\alpha+h^2tg^2\alpha\,\dot{\varphi}^2+\dot{h}^2)$$

$$=\frac{1}{2}m\left(\dot{h}^2tg^2\alpha+h^2tg^2\alpha\frac{l^2}{m^2}\frac{1}{(h\,tg\alpha)^4}+\dot{h}^2\right)$$

$$=\frac{1}{2}m\left(\dot{h}^2tg^2\alpha+\frac{l^2}{m^2tg^2\alpha}\frac{1}{h^2}+\dot{h}^2\right)$$

Además $E_0 = \frac{1}{2} m v_0^2 + mgh_0$

$$\Rightarrow E_0 = \frac{1}{2} m\left(\dot h^2 tg^2\alpha + \frac{\ell^2}{m^2 tg^2\alpha}\frac{1}{h^2} + \dot h^2\right) + mgh$$

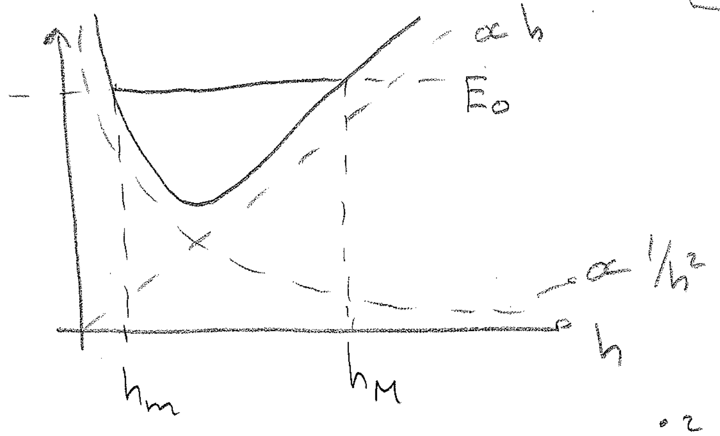
$\ell^2 = (m\,tg\alpha)^2 h_0^2 v_0^2$

$$\frac{1}{2} m v_0^2 + mgh_0 = \frac{1}{2} m \dot h^2 (1+tg^2\alpha) + \frac{h_0^2}{h^2} v_0^2 \frac{1}{2} m + mgh$$

$$\Rightarrow \dot h^2 = \frac{2}{m}\left(\frac{1}{1+tg^2\alpha}\right)\left[\frac{1}{2} m v_0^2\left(1-\frac{h_0^2}{h^2}\right) + mg(h_0-h)\right]$$

c) valores extremos $\dot h(h_e) = 0$

tengo $E_0 = \frac{1}{2} m (1+tg^2\alpha)\dot h^2 + \underbrace{\frac{1}{2}\frac{\ell^2}{m\,tg^2\alpha}\frac{1}{h^2} + mgh}_{U_{eff}(h)}$



tengo trayectorias acotadas

puntos de retroceso verifican $\dot h^2 = 0 \Rightarrow 0 = \frac{1}{2} m v_0^2\left(1-\frac{h_0^2}{h^2}\right) + mg(h_0-h)$

$$0 = \frac{1}{2} m \frac{v_0^2}{h^2}\underbrace{\left(h^2-h_0^2\right)}_{(h-h_0)(h+h_0)} + mg(h_0-h) \Rightarrow 0 = \frac{(h-h_0)}{h^2}\left[\frac{1}{2} m v_0^2 (h+h_0) - mgh^2\right]$$

$\Rightarrow \boxed{h=h_0}$ punto de retroceso y $-mgh^2 + \frac{1}{2} m v_0^2 h + \frac{1}{2} m v_0^2 h_0 = 0$

$$h = \frac{-\frac{1}{2} m v_0^2 \pm \sqrt{\left(\frac{1}{2} m v_0^2\right)^2 + 4mg\frac{1}{2} m v_0^2 h_0}}{-2mg}$$

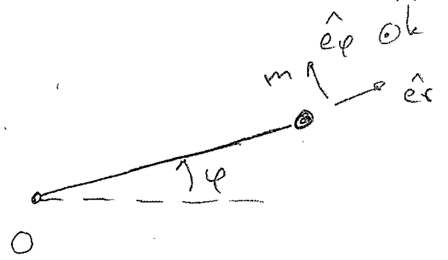
el signo − no es válido

$$h = \frac{v_0^2}{4g} \mp \frac{1}{2gm}\sqrt{\left(\frac{1}{2} m v_0^2\right)^2\left(1+8\frac{gh_0}{v_0^2}\right)} = \frac{v_0^2}{4g}\left(1 \mp \sqrt{1+8\frac{h_0 g}{v_0^2}}\right)$$

$$\Rightarrow \boxed{h = \frac{v_0^2}{4g}\left(1+\sqrt{1+8\frac{h_0 g}{v_0^2}}\right)}$$

V

Ejercicio 2



el hilo se acorta con v_0 constante

$\Rightarrow \dot{r} = -v_0$ cte.

a $t=0$ $\vec{r}(0) = d\hat{e}_r$ y $\dot{\varphi}(0) = \omega$

a) $\vec{r} = r\hat{e}_r$

$\vec{v} = \dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi$

$\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\,\hat{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{e}_\varphi$

Fuerzas $\vec{T} = -T\hat{e}_r$

$$\vec{F}_N = m\vec{a} \rightarrow -T\hat{e}_r = m(\ddot{r} - r\dot{\varphi}^2)\,\hat{e}_r + m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{e}_\varphi$$

$\rightarrow \hat{e}_r)$ $T = m(r\dot{\varphi}^2 - \ddot{r}) = mr\dot{\varphi}^2$ pues $\ddot{r} = 0$

$\hat{e}_\varphi)$ $0 = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = 0$ conservación de $\vec{L}_0$

$$\vec{L}_0 = \vec{r} \wedge m\vec{v} = r\hat{e}_r \wedge m(\dot{r}\hat{e}_r + r\dot{\varphi}\hat{e}_\varphi) = mr^2\dot{\varphi}\hat{k}$$

$$\ell = mr^2\dot{\varphi} = md^2\omega \rightarrow \boxed{\dot{\varphi} = \omega\frac{d^2}{r^2}}$$

$$\Rightarrow T = mr\frac{(\omega d^2)^2}{r^4} \rightarrow \boxed{T = m\frac{(\omega d^2)^2}{r^3}}$$

b) Trabajo desde d hasta b por la fuerza T

TM del trabajo y la energía $\dfrac{dT}{dt} = \dfrac{dW}{dt}$

También se puede:
$dW_{\vec{T}} = \vec{T}\cdot d\vec{r}$
$d\vec{r} = dr\hat{e}_r + rd\varphi\hat{e}_\varphi$
$W = \int_d^b -T\hat{e}_r\cdot(dr\hat{e}_r + rd\varphi\hat{e}_\varphi)$

$$\Rightarrow T(b) - T(d) = W_{d\rightarrow b}$$

$$T(d) = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(v_0^2 + d^2\omega^2)$$

$$T(b) = \frac{1}{2}m\vec{v}_b^2 = \frac{1}{2}m\left(v_0^2 + b^2\omega^2\frac{d^4}{b^4}\right) = \frac{1}{2}mv_0^2 + \frac{1}{2}m\omega^2d^2\left(\frac{d^2}{b^2}\right)$$

$$\Rightarrow W_{b\rightarrow d} = T(b) - T(d) = \frac{1}{2}m\omega^2d^2\left(\frac{d^2}{b^2} - 1\right) = \frac{1}{2}m\omega^2\frac{d^2}{b^2}(d^2 - b^2)$$

c) Trayectoria de la partícula.

$$\vec{T} = -\frac{m(\omega d^2)^2}{r^3}\hat{e}_r \qquad \text{uso Ecs. de Binet } u(\theta)=\frac{1}{r}$$

$$\Rightarrow \vec{T} = -A u^3 \qquad a_r = -\frac{l^2 u^2}{m^2}[u+u'']$$

$$\bar{a}_r = \frac{\vec{T}}{m} \qquad -\frac{A}{m}u^3 = -\frac{l^2 u^2}{m^2}[u+u'']$$

$$\rightarrow \frac{mA}{l^2}u = u+u'' \quad \rightarrow u''+u\left[1-\frac{mA}{l^2}\right]=0$$

$$l^2=(md^2\omega)^2 \qquad \frac{mA}{l^2} = \frac{m(\omega d^2)^2 m}{(md^2\omega)^2} = 1$$

$$\Rightarrow u''=0 \quad \rightarrow u(\varphi) = \underbrace{\alpha\varphi}_{} + \beta$$

podría ser también $\alpha(\varphi-\varphi_0)$ pero tomo $\varphi_0=0$

$$u(0) = \frac{1}{r(0)} = \frac{1}{d} = \beta$$

$$\dot{r} = -V_0 \qquad \dot{r} = \frac{d}{dt}(r) = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} =$$

$$\frac{dr}{du} = -\frac{1}{u^2} \qquad \Rightarrow \dot{r} = -\frac{\dot{\varphi}}{u^2}u' \quad \Rightarrow \quad \dot{r} = -\omega d^2 u'$$

$$\dot{\varphi} = \frac{\omega d^2}{r^2} = \omega d^2 u^2$$

$$\text{y } \dot{r} = -V_0 \Rightarrow V_0 = \omega d^2 u'$$

$$u' = \alpha \quad \rightarrow V_0 = \omega d^2 \alpha \quad \rightarrow \alpha = \frac{V_0}{\omega d^2}$$

$$\Rightarrow u(\varphi) = \frac{V_0}{\omega d^2}\varphi + \frac{1}{d} = \frac{V_0\varphi + \omega d}{\omega d^2} = \frac{1}{r(\varphi)} \Rightarrow \boxed{r(\varphi) = \frac{\omega d^2}{\omega d + V_0\varphi}}$$