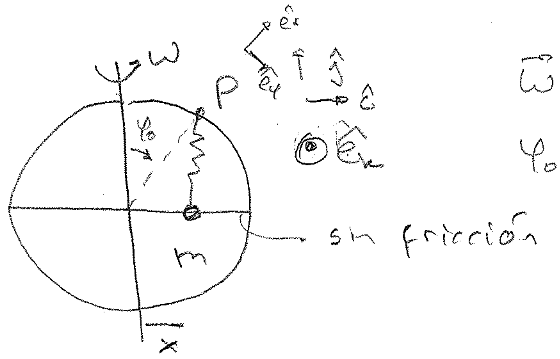
Problemas de clase: Práctico 3

1



$\vec\omega = \omega\hat j$ cte   resorte k, long. natural nula

φ_0 cte.

a) $\vec r = x\hat i$

$\dot{\hat i} = \vec\omega \wedge \hat i = -\omega\hat e_k$

$\dot{\hat j} = \vec\omega \wedge \hat j = 0$

$\dot{\hat e}_k = \vec\omega \wedge \hat e_k = \omega\hat i$

$\vec v = \dot x\hat i + x\dot{\hat i}$

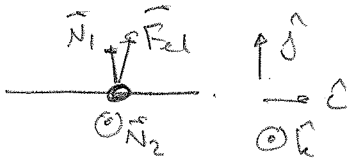
$\vec v = \dot x\hat i - \omega x\hat e_k$

$\vec a = \ddot x\hat i + \dot x\dot{\hat i} - \omega\dot x\hat e_k - \omega x\dot{\hat e}_k$

$\vec a = (\ddot x - \omega^2 x)\hat i - 2\omega\dot x\hat e_k$

$\vec r_P = R\hat e_r = R\cos\varphi_0\,\hat j + R\sin\varphi_0\,\hat i$

Fuerzas



$\vec N = N_1\hat j + N_2\hat e_k$

$\vec F_{el} = -k(\vec r - \vec r_P)$

$= -k\left[(x - R\sin\varphi_0)\hat i - R\cos\varphi_0\,\hat j\right]$

$\vec F_N = (N_1 + kR\cos\varphi_0)\hat j - k(x - R\sin\varphi_0)\hat i + N_2\hat e_k = m\vec a$

$m(\ddot x - \omega^2 x) = -k(x - R\sin\varphi_0)$ → ec. de mov.

$0 = N_1 + kR\cos\varphi_0$ → $\boxed{N_1 = -kR\cos\varphi_0}$

$-2\omega\dot x m = N_2$

$$\boxed{\ddot x + \left(\frac{k}{m} - \omega^2\right)x = \frac{k}{m}R\sin\varphi_0}$$

b) $\ddot{x} + \left(\frac{k}{m} - \omega^2\right)x - A = 0$ con $A = \frac{k}{m} R \sin\varphi_0$

Es de la forma $\ddot{x} + f(x) = 0$ No conservativo Preintegrable.

Se puede

$$f(x) = \frac{dF(x)}{dx} \; ??$$

$$F(x) = +\left(\frac{k}{m} - \omega^2\right)\frac{x^2}{2} - Ax$$

por similitud con los sistemas conservativos busco mínimos de $F(x)$

x_{eq} t.q $\left.\frac{dF(x)}{dx}\right|_{x_{eq}} = f(x_{eq}) = 0$ y $\left.\frac{d^2F(x)}{dx^2}\right|_{x_{eq}} > 0$

si $\omega^2 = \frac{k}{m}$ → $F(x)$ No tiene extremos.

$$x_{eq} = \frac{A}{\left(\frac{k}{m} - \omega^2\right)}$$

$$\frac{d^2F}{dx^2} = \left(\frac{k}{m} - \omega^2\right)$$

→

si $\omega^2 < \frac{k}{m}$ $x_{eq} = \frac{A}{\left|\frac{k}{m} - \omega^2\right|}$ estable $\left(\left.\frac{d^2F}{dx^2}\right|_{x_{eq}} > 0\right)$

si $\omega^2 > \frac{k}{m}$ $x_{eq} = \frac{-A}{\left|\omega^2 - \frac{k}{m}\right|}$ inestable $\left(\frac{d^2F}{dx^2} < 0\right)$

ii) Cond. inicial. $x(0) = 0$ y $\dot{x}(0) = 0$

$$\ddot{x} + \left(\frac{k}{m} - \omega^2\right)x - A = 0 \qquad x = x_h(t) + x_p(t)$$

$$x_p = \frac{A}{\frac{k}{m} - \omega^2}$$

$$x_h(t) = C e^{\lambda t}$$

$$\lambda^2 + \left(\frac{k}{m} - \omega^2\right) = 0 \rightarrow \lambda = \pm\sqrt{\omega^2 - \frac{k}{m}} \quad \text{si } \omega^2 > \frac{k}{m}$$

$$\lambda = \pm\sqrt{\omega^2 - \frac{k}{m}} = \pm\beta$$

$$x_h(t) = C e^{\beta t} + D e^{-\beta t}$$

$$x(t) = C e^{\beta t} + D e^{-\beta t} + x_p \qquad \dot{x}(0) = 0 = \beta(C - D) = 0$$

$C = D$

- $\dot{x}(t) = 2C\cosh(\beta t) + x_p$

$0 = x(0) = 2C + x_p \rightarrow C = -\frac{x_p}{2}$

$\Rightarrow x(t) = x_p\left(1 - \cosh(\beta t)\right) = \frac{A}{\frac{k}{m} - \omega^2}\left(1 - \cosh(\beta t)\right) = \frac{A}{\omega^2 - \frac{k}{m}}\left(\cosh(\beta t) - 1\right)$

si $\omega^2 < \frac{k}{m} \rightarrow \lambda = \pm j\sqrt{\frac{k}{m} - \omega^2} = \pm j\gamma$

$x_h(t) = Ce^{j\gamma t} + De^{-j\gamma t}$

$x(t) = Ce^{j\gamma t} + De^{-j\gamma t} + x_p$

$\dot{x}(0) = 0 \rightarrow j\gamma(C - D) = 0 \rightarrow C = D \Rightarrow e^{j\gamma t} + e^{-j\gamma t} = 2\cos(\gamma t)$

$x(t) = 2C\cos(\gamma t) + x_p$

$x(0) = 0 = 2C + x_p = 0 \rightarrow C = -\frac{x_p}{2}$

$x(t) = x_p\left(1 - \cos(\gamma t)\right) = \frac{A}{\frac{k}{m} - \omega^2}\left(1 - \cos(\gamma t)\right)$

si $\omega^2 = \frac{k}{m} \rightarrow \ddot{x} = A \rightarrow x(t) = A\frac{t^2}{2}$

$\cosh(\beta t) \xrightarrow[t \to \infty]{} \infty$ movimiento NO acotado

$t^2 \xrightarrow[t \to \infty]{} \infty$ Movimiento NO acotado

$|\cos(\gamma t)| \leq 1 \ \forall t$ movimiento Acotado $\Rightarrow$ caso $\omega^2 < \frac{k}{m}$ es acotado

para movimiento acotado $x(t) < R \ \forall t \Rightarrow x_{máx} < R$

$x(t) = x_p(1 - \cos(\gamma t)) \rightarrow x_{máx} = 2x_p \rightarrow \boxed{x_p < \frac{R}{2}}$

$$x_p = \frac{kR\sin\varphi_0}{k-m\omega^2} < \frac{R}{2} \rightarrow \frac{k}{k-m\omega^2}\sin\varphi_0 < \frac{1}{2}$$

$$\frac{k}{k-m\omega^2} > 1 \quad \text{pues } \omega^2 < \frac{k}{m} \Rightarrow \sin\varphi_0 < \frac{1}{2}$$

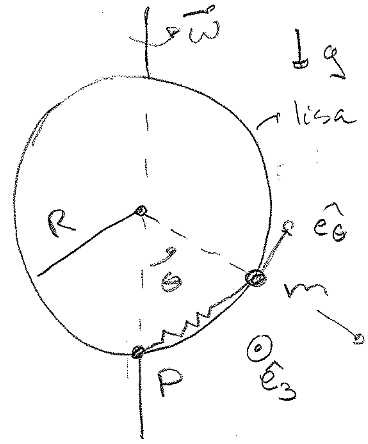
$\rightarrow$ tengo un límite para φ_0

desarrollemos $x_p < \frac{R}{2}$ $\quad 2k\sin\varphi_0 < k - m\omega^2$

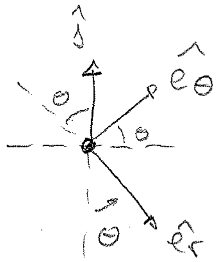
$$\rightarrow m\omega^2 < k - 2k\sin\varphi_0$$

$$0 < \omega^2 < \frac{k}{m}(1 - 2\sin\varphi_0) \rightarrow \sin\varphi_0 < \frac{1}{2}$$

$\boxed{2.}$ $\boxed{IV}$



resorte $k = \frac{mg}{R}$ $\quad l_0 = 0$



$$\vec{\omega} = \omega \hat{j} + \dot{\theta} \hat{e}_3$$

$$\vec{\omega} = -\omega\cos\theta\, \hat{e}_r + \omega\sin\theta\, \hat{e}_\theta + \dot{\theta}\hat{e}_3 \qquad \hat{j} = -\cos\theta\, \hat{e}_r + \sin\theta\, \hat{e}_\theta$$

$$\vec{r} = R\,\hat{e}_r$$

$$\vec{v} = R\,\dot{\hat{e}}_r$$

$$\dot{\hat{e}}_r = \vec{\omega} \wedge \hat{e}_r = -\omega\sin\theta\, \hat{e}_3 + \dot{\theta}\, \hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = \vec{\omega} \wedge \hat{e}_\theta = -\omega\cos\theta\, \hat{e}_3 - \dot{\theta}\, \hat{e}_r$$

$$\dot{\hat{e}}_3 = \vec{\omega} \wedge \hat{e}_3 = +\omega\cos\theta\, \hat{e}_\theta + \omega\sin\theta\, \hat{e}_r$$

$$\vec{v} = -R\omega\sin\theta\, \hat{e}_3 + R\dot{\theta}\, \hat{e}_\theta$$

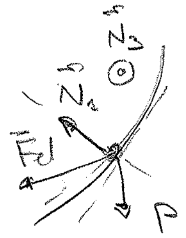
$$\vec{a} = -R\omega\dot{\theta}\cos\theta\, \hat{e}_3 - R\omega\sin\theta\, \dot{\hat{e}}_3 + R\ddot{\theta}\, \hat{e}_\theta + R\dot{\theta}\, \dot{\hat{e}}_\theta$$

$$= -R\omega\dot{\theta}\cos\theta\, \hat{e}_3 - R\omega^2\sin\theta\cos\theta\, \hat{e}_\theta - R\omega^2\sin^2\theta\, \hat{e}_r + R\ddot{\theta}\, \hat{e}_\theta$$

$$+ R\dot{\theta}\left(-\omega\cos\theta\, \hat{e}_3 - \dot{\theta}\, \hat{e}_r\right)$$

$$\vec{a} = -\left(R\omega^2\sin^2\theta + R\dot{\theta}^2\right)\hat{e}_r + \left(R\ddot{\theta} - R\omega^2\sin\theta\cos\theta\right)\hat{e}_\theta - 2R\omega\dot{\theta}\cos\theta\, \hat{e}_3$$

Fuerzas



$$\vec{N} = -N_1\, \hat{e}_r + N_3\, \hat{e}_3$$

$$\vec{P} = -mg\,\hat{j} = mg\cos\theta\, \hat{e}_r - mg\sin\theta\, \hat{e}_\theta$$

$$\vec{F}_{el} = -k\left(\vec{r} - \vec{r}_p\right) = -k\left(R\hat{e}_r + R\hat{j}\right)$$

$$\vec{r}_p = -R\,\hat{j}$$

$$\vec{F}_{el} = -k\left(R(1-\cos\theta)\, \hat{e}_r + R\sin\theta\, \hat{e}_\theta\right)$$

$$\vec{F}_N = \left(mg\cos\theta - N_1 - kR(1-\cos\theta)\right)\hat{e}_r - \left(mg\sin\theta + kR\sin\theta\right)\hat{e}_\theta + N_3\hat{e}_3$$

$$k = \frac{mg}{R} \Rightarrow \vec{F}_N = \left(mg\left(2\cos\theta - 1\right) - N_1\right)\hat{e}_r - 2mg\sin\theta\, \hat{e}_\theta + N_3\, \hat{e}_3$$

2da Ley $\vec{F}_N = m\vec{a}$

$$\begin{cases} mg\left(2\cos\theta - 1\right) - N_1 = -m\left(R\dot{\theta}^2 + R\omega^2\sin^2\theta\right) \\ -2mg\sin\theta = m\left(R\ddot{\theta} - R\omega^2\sin\theta\cos\theta\right) \longleftarrow \text{Ec. de movimiento} \\ N_3 = -2mR\omega\dot{\theta}\cos\theta \end{cases}$$

V

b) $\ddot{\theta} - \omega^2 \sin\theta\cos\theta + \frac{2g}{R}\sin\theta = 0$

$$\ddot{\theta} + \left(\frac{2g}{R} - \omega^2\cos\theta\right)\sin\theta = 0$$

Ec. de la forma $\ddot{\theta} + f(\theta) = 0 \rightarrow$ busco $F(\theta)$ t.q. $\frac{dF}{d\theta} = f$

los mínimos de $F(\theta)$ son ptos de equilibrio

$$\left.\frac{dF}{d\theta}\right|_{\theta_{eq}} = 0 \rightarrow \left.\left(\frac{2g}{R} - \omega^2\cos\theta\right)\sin\theta\right|_{\theta_{eq}} = 0 \rightarrow \sin\theta_{eq} = 0 \rightarrow \theta_{eq} = \begin{cases} 0 \\ \pi \end{cases}$$

$\hookrightarrow \frac{2g}{R} - \omega^2\cos\theta_{eq} = 0$

$\cos\theta_{eq} = \frac{2g}{\omega^2 R}$ $\exists$ sii $2g < \omega^2 R$

Estabilidad ? $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_{eq}} > 0$ estable.

$$\frac{d^2F}{d\theta^2} = \frac{df}{d\theta} = \cos\theta\left(\frac{2g}{R} - \omega^2\cos\theta\right) + \omega^2\sin^2\theta$$

$$= \frac{2g}{R}\cos\theta + \omega^2\left(\sin^2\theta - \cos^2\theta\right)$$

$$= \frac{2g}{R}\cos\theta - \omega^2\cos^2\theta + \omega^2\sin^2\theta = \left(\frac{2g}{R} - \omega^2\cos\theta\right)\cos\theta + \omega^2\sin^2\theta$$

$\theta_{eq}^{(1)} = 0$ $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_{eq}^{(1)}} = \left(\frac{2g}{R} - \omega^2\right)$ estable sii $\omega^2 < \frac{2g}{R}$

$\theta_{eq}^{(2)} = \pi$ $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_{eq}^{(2)}} = -\left(\frac{2g}{R} + \omega^2\right)$ inestable siempre

$\theta_{eq}^{(3)}$ t.q. $\cos\theta_{eq}^{(3)} = \frac{2g}{\omega^2 R}$ $\left.\frac{d^2F}{d\theta^2}\right|_{\theta_{eq}^{(3)}} = \overbrace{\left(\frac{2g}{R} - \omega^2\cos\theta_{eq}^{(3)}\right)}^{= 0}\cos\theta_{eq}^{(3)} + \underbrace{\omega^2\sin^2\theta_{eq}^{(3)}}_{> 0}$

$\Rightarrow \theta_{eq}^{(3)}$ si $\exists$ es estable

c) $P_N = \vec{N}\cdot\vec{v} = (-N_1\hat{e}_r + N_3\hat{e}_3)\cdot(R\dot{\theta}\hat{e}_\theta - R\omega\sin\theta\,\hat{e}_3)$

$= -N_3 R\omega\sin\theta \Rightarrow \boxed{P_N = 2m\omega^2 R^2\dot{\theta}\sin\theta\cos\theta}$