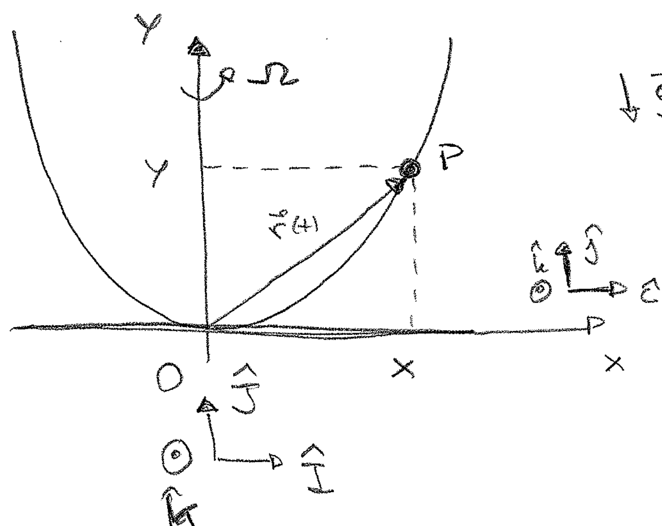
Problemas de Clase: Práctico 2

Ejercicio 1:



$\downarrow \vec{g}$

guía lisa parabólica $y = \alpha x^2$

Ω cte.

$S\{O, \hat{I}, \hat{J}, \hat{K}\}$ Absoluto $\qquad$ Nota $\hat{J} \equiv \hat{j}$

$S_1\{O, \hat{i}, \hat{j}, \hat{k}\}$ auxiliar $\qquad \vec{\omega}_{S_1} = \Omega \hat{j}$

derivo directo $\qquad \vec{r} = x\hat{i} + y\hat{j} = x\hat{i} + \alpha x^2 \hat{j}$

$\hookrightarrow$ guía parabólica

$$\vec{v} = \dot{x}\hat{i} + x\dot{\hat{i}} + 2\alpha x\dot{x}\hat{j} + \alpha x^2 \dot{\hat{j}}$$

$$\vec{v} = \dot{x}\hat{i} - \Omega x\hat{k} + 2\alpha x\dot{x}\hat{j}$$

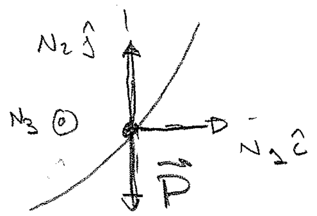
$$\dot{\hat{i}} = \vec{\omega}_{S_1} \wedge \hat{i} = \Omega\hat{j} \wedge \hat{i} = -\Omega\hat{k}$$

$$\dot{\hat{j}} = \vec{\omega}_{S_1} \wedge \hat{j} = \Omega\hat{j} \wedge \hat{j} = 0$$

$$\dot{\hat{k}} = \vec{\omega}_{S_1} \wedge \hat{k} = \Omega\hat{j} \wedge \hat{k} = \Omega\hat{i}$$

$$\vec{a} = \ddot{x}\hat{i} + \dot{x}\dot{\hat{i}} - \Omega\dot{x}\hat{k} - \Omega x\dot{\hat{k}} + (2\alpha\dot{x}^2 + 2\alpha x\ddot{x})\hat{j}$$

$$\vec{a} = (\ddot{x} - \Omega^2 x)\hat{i} - 2\Omega\dot{x}\hat{k} + (2\alpha\dot{x}^2 + 2\alpha x\ddot{x})\hat{j}$$



Dinámica: Fuerzas $\rightarrow$ peso $\vec{P} = -mg\hat{j}$

Para la Normal supongo $\vec{N} = N_1\hat{i} + N_2\hat{j} + N_3\hat{k}$

$$\vec{F}_N = N_1\hat{i} + (N_2 - mg)\hat{j} + N_3\hat{k} \qquad \vec{F}_N = m\vec{a}$$

$$N_1\hat{i} + (N_2 - mg)\hat{j} + N_3\hat{k} = m(\ddot{x} - \Omega^2 x)\hat{i} - 2m\Omega\dot{x}\hat{k} + m(2\alpha\dot{x}^2 + 2\alpha x\ddot{x})\hat{j}$$

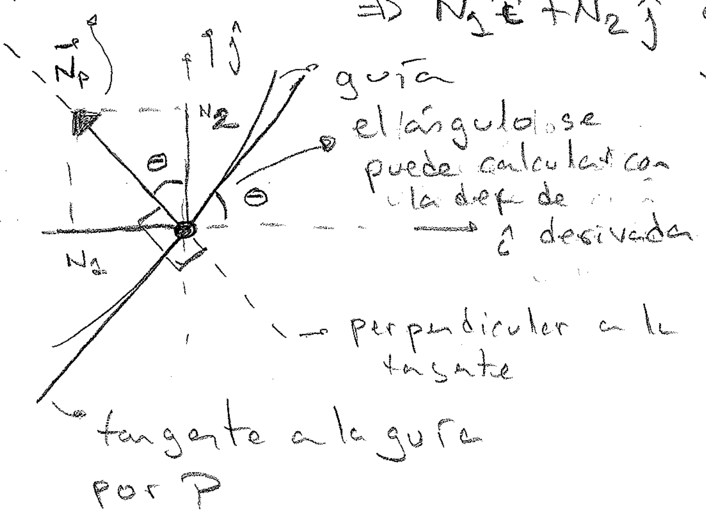
$$\Rightarrow \begin{cases} N_1 = m(\ddot{x} - \Omega^2 x) \\ N_2 - mg = m2\alpha(\dot{x}^2 + x\ddot{x}) \\ N_3 = -2m\Omega\dot{x} \end{cases}$$

La normal debe estar en un plano perpendicular a la guía

$N_3\hat{k}$ es $\perp$ a la guía (es $\perp$ al plano definido por $\hat{i}, \hat{j}$)

$\vec{N}_p = N_1 \hat{\imath} + N_2 \hat{\jmath}$ debe ser perpendicular a la guía

supongo este sentido.



$\Rightarrow N_1 \hat{\imath} + N_2 \hat{\jmath}$ debe ser perpendicular al vector tangente a la guía que pasa por P.

de otra forma: la dirección de $N_1 \hat{\imath} + N_2 \hat{\jmath}$ debe ser $\perp$ a la dirección tangente a la guía en el punto P

con esta construcción $N_1 = -N_p \sin\theta$

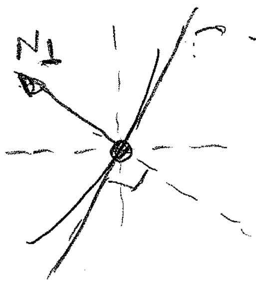
$$N_2 = N_p \cos\theta$$

por la definición de derivada $tg(\theta) = \frac{dy}{dx}$

$$\Rightarrow tg(\theta) = \frac{dy}{dx} = \frac{d}{dx}(\alpha x^2) = 2x\alpha$$

Obs.: En este caso, tomamos el camino de escribir $\vec{N}$ de forma general y después imponer que debe ser perpendicular a la guía.

Otra opción sería escribir $\vec{N} = \vec{N}_\perp + N_3 \hat{k}$ con $\vec{N}_\perp$ es $\perp$ a la guía y está en el plano $\{\hat{\imath}, \hat{\jmath}\}$. Hacer la figura



tangente a la guía y buscar las proyecciones para aplicar la igualdad en la 2^{da} Ley de Newton.

Volvemos a la 2^{da} Ley
$$N_1 = m(\ddot{x} - \Omega^2 x)$$
$$N_2 = mg + 2\alpha m(\dot{x}^2 + x\ddot{x})$$

$$N_1 = -N_p \sin\theta$$
$$N_2 = N_p \cos\theta$$
$$tg(\theta) = 2\alpha x$$

$N_1 = m(\ddot{x} - \Omega^2 x)$ las divido entre si $\frac{N_1}{N_2} = -tg(\theta)$

$N_2 = mg + 2\alpha m(\dot{x}^2 + x\ddot{x})$

$$-tg(\theta) = \frac{m(\ddot{x} - \Omega^2 x)}{mg + 2\alpha m(\dot{x}^2 + x\ddot{x})} \Rightarrow$$

$$\boxed{-2\alpha x = \frac{m(\ddot{x} - \Omega^2 x)}{mg + 2\alpha m(\dot{x}^2 + x\ddot{x})}}$$ Ec. de mov.

c) En $t=0$ $x = x_0$ y $\dot{x}(0) = 0$

Uso la ec. de mov.

$$\Rightarrow -2\alpha x_0 = \frac{m(\ddot{x}(t=0) - \Omega^2 x_0)}{mg + 2\alpha m(x_0 \ddot{x}(t=0))}$$

para saber cual es el movimiento relativo próximo a $t=0$ estudio que ocurre con $\ddot{x}(t=0) = \ddot{x}_0$. Si $\ddot{x}_0 > 0$ se aleja del 0

$\ddot{x}_0 < 0$ se acerca a 0

despejo $\ddot{x}_0$

$$-2\alpha x_0 mg - (2\alpha)^2 m x_0^2 \ddot{x}_0 = m\ddot{x}_0 - m\Omega^2 x_0$$

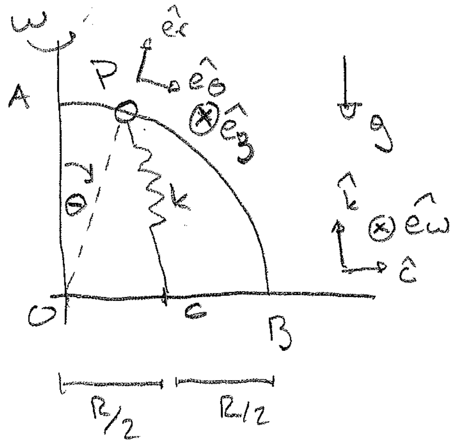
$$x_0(\Omega^2 - 2\alpha g) = \ddot{x}_0(1 + (2\alpha)^2 x_0^2)$$

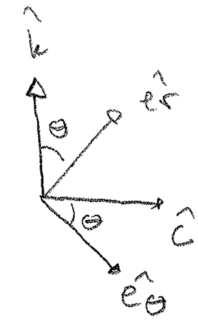
$$\Rightarrow \ddot{x}_0 = \frac{x_0(\Omega^2 - 2\alpha g)}{1 + (2\alpha)^2 x_0^2}$$

para que se aleje $\ddot{x}_0 > 0$

$\Rightarrow \boxed{\Omega^2 > 2\alpha g}$

Ejercicio 2.





$S_1\ \{O, \hat{\imath}, \hat{e}_\omega, \hat{k}\}$ rota siguiendo a la guía $\quad \bar{\omega}_{S1} = \omega \hat{k}$

$S_2\ \{O, \hat{e}_r, \hat{e}_\theta, \hat{e}_3\}$ obs $\hat{e}_3 \equiv \hat{e}_\omega \quad \bar{\omega}_{S2} = \omega\hat{k} + \dot{\theta}\hat{e}_\omega$

$$\dot{\hat{e}}_r = \bar{\omega}_{S2} \wedge \hat{e}_r$$
$$\hat{e}_r = \cos\theta\,\hat{k} + \sin\theta\,\hat{\imath}$$
$$\hat{e}_\theta = -\sin\theta\,\hat{k} + \cos\theta\,\hat{\imath}$$

$$\dot{\hat{e}}_\omega = (\omega\hat{k} \wedge \dot{\theta}\hat{e}_\omega) \wedge \hat{e}_\omega = -\omega\hat{\imath}$$

$$\vec{r} = R\hat{e}_r \qquad \vec{v} = R\dot{\hat{e}}_r$$

$$\dot{\hat{e}}_r = (\omega\hat{k} + \dot{\theta}\hat{e}_\omega)\wedge(\cos\theta\,\hat{k} + \sin\theta\,\hat{\imath})$$
$$\dot{\hat{e}}_r = \omega\sin\theta\,\hat{e}_\omega + \underbrace{\dot{\theta}\cos\theta\,\hat{\imath} - \dot{\theta}\sin\theta\,\hat{k}}_{\dot{\theta}\hat{e}_\theta}$$
$$\dot{\hat{e}}_\theta = (\omega\hat{k} + \dot{\theta}\hat{e}_\omega)\wedge(-\sin\theta\,\hat{k} + \cos\theta\,\hat{\imath})$$
$$= \omega\cos\theta\,\hat{e}_\omega \overbrace{- \dot{\theta}\sin\theta\,\hat{\imath} - \dot{\theta}\cos\theta\,\hat{k}}^{-\dot{\theta}\hat{e}_r}$$

$$\vec{v} = R\omega\sin\theta\,\hat{e}_\omega + R\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a} = R\omega\dot{\theta}\cos\theta\,\hat{e}_\omega + R\omega\sin\theta\,\dot{\hat{e}}_\omega + R\ddot{\theta}\,\hat{e}_\theta + R\dot{\theta}\,\dot{\hat{e}}_\theta$$

$$\vec{a} = R\omega\dot{\theta}\cos\theta\,\hat{e}_\omega + R\omega\sin\theta(-\omega\hat{\imath}) + R\ddot{\theta}\,\hat{e}_\theta + R\dot{\theta}\omega\cos\theta\,\hat{e}_\omega - R\dot{\theta}^2\hat{e}_r$$

$$\vec{a} = -R\dot{\theta}^2\hat{e}_r + R\ddot{\theta}\,\hat{e}_\theta - R\omega^2\sin\theta\,\underbrace{\hat{\imath}}_{\hat{\imath} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta} + 2R\omega\dot{\theta}\cos\theta\,\hat{e}_\omega$$

Dinámica: Fuerzas presentes guía lisa → Normal
→ No hay rozamiento
Peso, fuerza elástica

$\vec{N} = N\hat{e}_r + N_2\hat{e}_\omega$ debe estar en el plano ⊥ a la guía circular

N y N_2 pueden tener cualquier signo
(vínculo bilateral)

$\vec{P} = -mg\hat{k} \quad (\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) \Rightarrow \vec{P} = -mg\cos\theta\,\hat{e}_r + mg\sin\theta\,\hat{e}_\theta$

$\vec{F}_{el} = -k(\vec{r}_p - \vec{r}_c)$

V

$\vec{r}_P = R\hat{e_r}$

$\vec{r}_C = \frac{R}{2}\hat{c}$ $\Rightarrow \vec{F}_{el} = -k\left(R\hat{e_r} - \frac{R}{2}\hat{c}\right)$ $\Rightarrow$

$\hat{c} = \sin\theta\,\hat{e_r} + \cos\theta\,\hat{e_\theta}$

$$\vec{F}_{el} = -kR\left(\left(1 - \frac{1}{2}\sin\theta\right)\hat{e_r} - \frac{\cos\theta}{2}\hat{e_\theta}\right)$$

$$\vec{F}_N = N\hat{e_r} - mg\cos\theta\,\hat{e_r} + mg\sin\theta\,\hat{e_\theta} - kR\left(1 - \frac{1}{2}\sin\theta\right)\hat{e_r} + \frac{kR}{2}\cos\theta\,\hat{e_\theta}$$

$\vec{F}_N = m\vec{a}$ igualo componente a componente (Obs. debo pasar $\hat{c}$ a la base $\{\hat{e_r}, \hat{e_\theta}, \hat{e_\omega}\}$)

$\hat{e_r})$ $N - mg\cos\theta - \frac{kR}{2}(2 - \sin\theta) = -mR\dot{\theta}^2 - mR\omega^2\sin^2\theta$

$\hat{e_\theta})$ $\boxed{mg\sin\theta + \frac{kR}{2}\cos\theta = mR\ddot{\theta} - mR\omega^2\sin\theta\cos\theta}$

$\hat{e_\omega})$ $N_2 = 2mR\omega\dot{\theta}\cos\theta$

La ec. en $\hat{e_\theta}$ No contiene incognitas adicionales $\Rightarrow$ es la ec. de movimiento

$$\ddot{\theta} = \frac{k}{2m}\cos\theta + \omega^2\sin\theta\cos\theta + \frac{g}{R}\sin\theta$$

b) Velocidad absoluta al llegar al punto B.

$\vec{v} = R\omega\sin\theta\,\hat{e_\omega} + R\dot{\theta}\hat{e_\theta}$ debo calcular $\dot{\theta}$ y evaluar en $\theta = \pi/2$

preintegro la ec. de movimiento para llegar a $\dot{\theta} = f(\theta)$

multiplico por $\dot{\theta}$

$$\dot{\theta}\ddot{\theta} = \frac{k}{2m}\dot{\theta}\cos\theta + \omega^2\dot{\theta}\sin\theta\cos\theta + \frac{g}{R}\dot{\theta}\sin\theta$$

VI

para integrar en el tiempo uso q $\ddot{\theta} = \frac{d\dot{\theta}}{dt}$ y $\dot{\theta} = \frac{d\theta}{dt}$

$$\int \dot{\theta}\ddot{\theta}\,dt = \int_{t=0}^{t} \dot{\theta}\frac{d\dot{\theta}}{dt}\,dt = \int_{\dot{\theta}(t=0)}^{\dot{\theta}} \dot{\theta}\,d\dot{\theta} = \frac{1}{2}\dot{\theta}^2\Big|_{\dot{\theta}(0)}^{\dot{\theta}}$$

uso cambio de variable

el término de la derecha lo hago por sumandos:

$$\int_{t=0}^{t} \frac{k}{2m}\cos\theta \frac{d\theta}{dt}\,dt \underset{\text{cambio variable}}{=} \int_{\theta_0}^{\theta} \frac{k}{2m}\cos\theta\,d\theta = \frac{k}{2m}\sin\theta\Big|_{\theta_0}^{\theta}$$

$$\int_{t=0}^{t} \omega^2 \sin\theta\cos\theta \frac{d\theta}{dt}\,dt = \int_{\theta_0}^{\theta} \omega^2 \sin\theta\cos\theta\,d\theta = \frac{\omega^2}{2}\sin^2\theta\Big|_{\theta_0}^{\theta}$$

$$\int_{t=0}^{t} \frac{g}{R}\sin\theta \frac{d\theta}{dt}\,dt = \int_{\theta_0}^{\theta} \frac{g}{R}\sin\theta\,d\theta = -\frac{g}{R}\cos\theta\Big|_{\theta_0}^{\theta}$$

$$\Rightarrow \frac{1}{2}\left(\dot{\theta}^2(\theta) - \dot{\theta}^2(0)\right) = \frac{k}{2m}(\sin\theta - \sin\theta_0) + \frac{\omega^2}{2}(\sin^2\theta - \sin^2\theta_0) - \frac{g}{R}(\cos\theta - \cos\theta_0)$$

Cond. iniciales $\theta_0 = 0$ $\dot{\theta}(0) = 0$

$$\frac{1}{2}\dot{\theta}^2(\theta) = \frac{k}{2m}\sin\theta + \frac{\omega^2}{2}\sin\theta - \frac{g}{R}(\cos\theta - 1)$$

En B $\theta = \frac{\pi}{2}$ $\rightarrow \frac{1}{2}\dot{\theta}^2\left(\frac{\pi}{2}\right) = \frac{1}{2}\dot{\theta}_B^2 = \frac{k}{2m} + \frac{\omega^2}{2} + \frac{g}{R}$

$$\rightarrow \dot{\theta}_B^2 = \frac{k}{m} + \omega^2 + \frac{2g}{R} \rightarrow \vec{v}\left(\frac{\pi}{2}\right) = R\omega\hat{e_\omega} + R\dot{\theta}_B \overbrace{\hat{e_\theta}\left(\theta = \frac{\pi}{2}\right)}^{-\hat{k}}$$

c) $W = \int_A^B \vec{N}\cdot d\vec{r} = \int_A^B \vec{N}\cdot\vec{v}\,dt \rightarrow (N\hat{e_r} + N_2\hat{e_\omega})\cdot(R\omega\sin\theta\,\hat{e_\omega} + R\dot{\theta}\hat{e_\theta}) =$

$d\vec{r} = \vec{v}\,dt$

$= N_2 R\omega^2\sin\theta = 2mR^2\omega^2\sin\theta\cos\theta\,\dot{\theta}$

$$\Rightarrow W = \int_A^B 2mR^2\omega^2\sin\theta\cos\theta\,\underbrace{\dot{\theta}\,dt}_{\frac{d\theta}{dt}dt} \underset{\text{cambio de variable}}{=} \int_{\theta=0}^{\pi/2} 2mR^2\omega^2\sin\theta\cos\theta\,d\theta = mR^2\omega^2\sin^2\theta\Big|_{\theta=0}^{\theta=\pi/2}$$

$$\boxed{W = mR^2\omega^2}$$