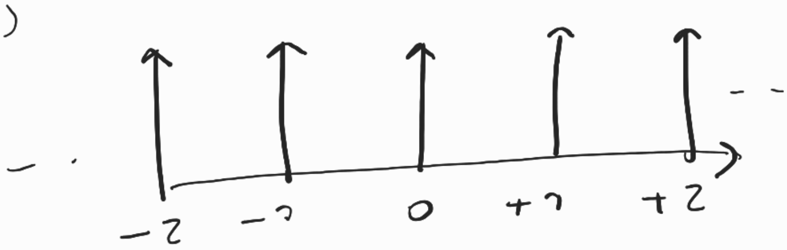
Transformadas de interés:

▣ <u>Función Peine de Dirac</u> ('comb'):

$$Ш(x) = \sum_{-\infty}^{+\infty} \delta(x-n)$$

('shah')



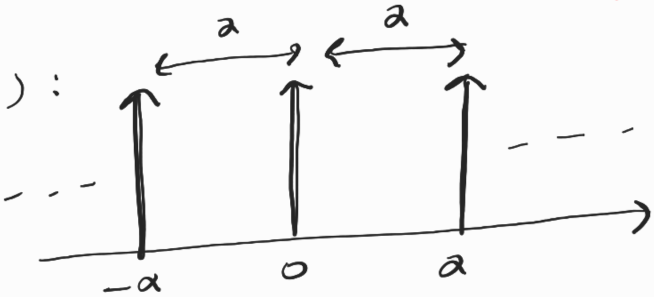
• Si el intervalo no es unidad:

$$Ш(x/a) \equiv \sum_{-\infty}^{+\infty} \delta(x/a - n) = \sum_{-\infty}^{+\infty} \delta\left(\frac{x-an}{a}\right) =$$

(ejercicio)

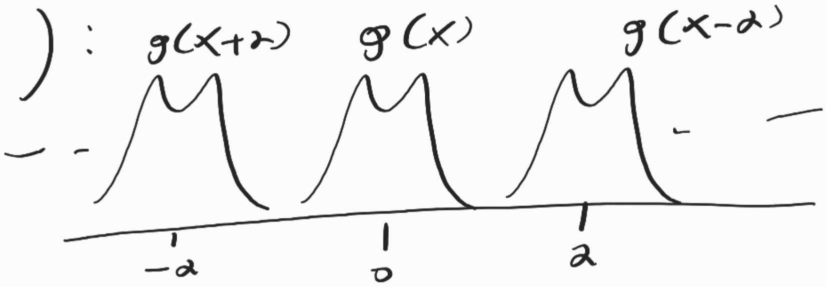
$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

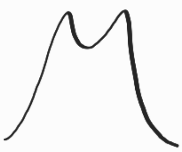
$$a \sum_{-\infty}^{+\infty} \delta(x-an):$$



<u>Obs:</u> como $g(x) * \delta(x-n) = g(x-n)$:

$$\Rightarrow g * \left(\frac{1}{a} Ш(x/a)\right):$$





genera un tren de réplicas!

• Transformada del peine:

$$Ш(x) = \sum_{n=-\infty}^{+\infty} a_n e^{j2\pi n x}, \quad a_n = \int_{-1/2}^{1/2} Ш(x) e^{-j2\pi n x} dx$$

$$a_n = \int_{-1/2}^{1/2} \left(\sum_{l=-\infty}^{+\infty} \delta(x-l) \right) e^{-j2\pi nx}\, dx = 1$$

só contribuye $l=0$ al intervalo $(-1/2, 1/2)$

$$\Rightarrow \text{III}(x) = \sum_{-\infty}^{+\infty} e^{j2\pi nx};$$

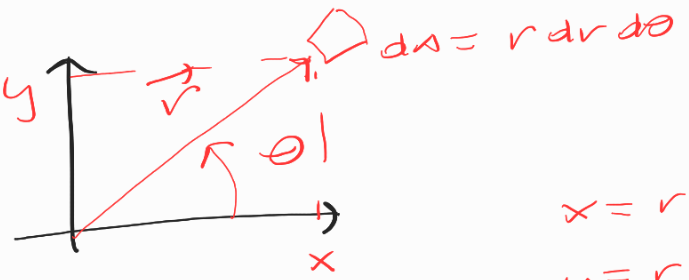
$$\mathcal{F}\{\text{III}(x)\} = \mathcal{F}\left[\sum_{-\infty}^{+\infty} \delta(x-n)\right] =$$

$$\sum_{-\infty}^{+\infty} \underbrace{\mathcal{F}\{\delta(x-n)\}}_{\int_{-\infty}^{+\infty} \delta(x-n)\, e^{-j2\pi f_x x}\, dx \,=\, e^{-j2\pi f_x n}} = \sum_{-\infty}^{+\infty} e^{j2\pi f_x n} = \text{III}(f_x)$$

Idem: $\text{III}(x/a) \xrightarrow[TF]{} a\,\text{III}(\underbrace{a f_x}_{f_x/(1/a)})$

- Transformada de Fourier en coordenadas polares:

$$G(f_x, f_y) = \iint_{-\infty}^{+\infty} g(x,y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy$$



$x = r\cos\theta$

$y = r\,\text{sen}\,\theta$

Idem: $f_x = \rho\cos\phi$

$f_y = \rho\,\text{sen}\,\phi$

$$\Rightarrow G(\rho,\phi) = \int_0^\infty \int_0^{2\pi} dr\, d\theta\; r\, g(r,\theta)\, e^{-j2\pi\rho r \cos(\phi-\theta)}$$

$g(r,\theta)$ univaluada: $g(r,\theta+2\pi) = g(r,\theta)$

$$\Rightarrow g(r,\theta) = \sum_{-\infty}^{+\infty} \underbrace{g_n(r)}_{} e^{jn\theta}$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)\, e^{-jn\theta}\, d\theta$$

$$G(\rho,\phi) = \sum_{n=-\infty}^{+\infty} \int_0^\infty dr\, r\, g_n(r) \int_0^{2\pi} d\theta\, e^{-j2\pi\rho r \underbrace{\cos(\phi-\theta)}_{=\cos(\theta-\phi)} + jn\theta}$$

Bessel en forma integral:

$$J_n(z) = \frac{1}{2\pi}\int_\alpha^{\alpha+2\pi} e^{j(n\beta - z\cos\beta)}\, d\beta$$

$$\beta = \theta - \phi + \frac{3\pi}{2}$$

$$\int_0^{2\pi} d\theta\, e^{-j2\pi\rho r \cos(\theta-\phi) + jn\theta} =$$

$$= e^{jn\,3/2\pi + jn\phi} \int_{-\phi-3\pi/2}^{2\pi-\phi-3\pi/2} d\beta\;\; e^{j(n\beta - 2\pi\rho r \cos\beta)}$$

$$= (-j)^n e^{jn\phi}\, 2\pi\, J_n(2\pi\rho r)$$

$\Rightarrow G(\rho,\phi) =$

Transformada de Hankel de orden n

$$\sum_{-\infty}^{+\infty} (-j)^n e^{jn\phi}\, \overbrace{2\pi \int_0^\infty r\, g_n(r)\, J_n(2\pi \rho r)\, dr}$$

con: $g_n(r) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g(r,\theta)\, e^{-jn\theta}\, d\theta$

Transformada inversa:

$$g(r,\theta) = \sum_{-\infty}^{+\infty} (j)^n e^{jn\theta}\, 2\pi \int_0^\infty \rho\, G_n(\rho)\, J_n(2\pi r \rho)\, d\rho$$

con: $G_n(\rho) = \dfrac{1}{2\pi} \displaystyle\int_0^\infty G(\rho,\phi)\, e^{-jn\phi}\, d\phi$

- <u>Funciones con simetría azimutal</u>:

no hay variación en θ : $g_n(r) = \begin{cases} g_0(r) & n=0 \\ 0 & n\neq 0 \end{cases}$

$$\Rightarrow G_0(\rho) = 2\pi \int_0^\infty r\, g_0(r)\, J_0(2\pi\rho r)\, dr$$ Hankel orden 0

$$g_0(r) = 2\pi \int_0^\infty \rho\, G_0(\rho)\, J_0(2\pi r\rho)\, d\rho$$

<u>Eg</u>: función círculo : $\mathrm{circ}(r/R) = \begin{cases} 1, & r \leq R \\ 0, & r > R \end{cases}$

$\Rightarrow$

$$\mathcal{F}\{\mathrm{circ}(r/R)\} = 2\pi \int_0^R r\, J_0(2\pi\rho r)\, dr$$

usando que: $\int_0^x x'\, J_0(x')\, dx' = x\, J_1(x)$

$$= 2\pi \int_0^{2\pi\rho R} \left(\frac{x}{2\pi\rho}\right) J_0(x)\, dx / (2\pi\rho)$$

$\uparrow$ $x = 2\pi\rho r$; $dx = 2\pi\rho\, dr$

$$= 2\pi \frac{1}{(2\pi\rho)^{\cancel{2}}R} R\, (\cancel{2\pi\rho} R)\, J_1(2\pi\rho R) =$$

$$= \pi R^2 \frac{2 J_1(2\pi\rho R)}{(2\pi\rho R)}, \quad \mathrm{Besinc}\,(2\pi\rho R)$$

sinc de Bessel o Jinc

$$\mathrm{Besinc}\,(x) = 2\frac{J_1(x)}{x}$$

Obs: $\lim_{x\to 0} \frac{J_1(x)}{x} = \lim_{x\to 0} \frac{J_1'(x)}{1}$

$\uparrow$ L'Hospital

• Recurrencia de las funciones de Bessel:

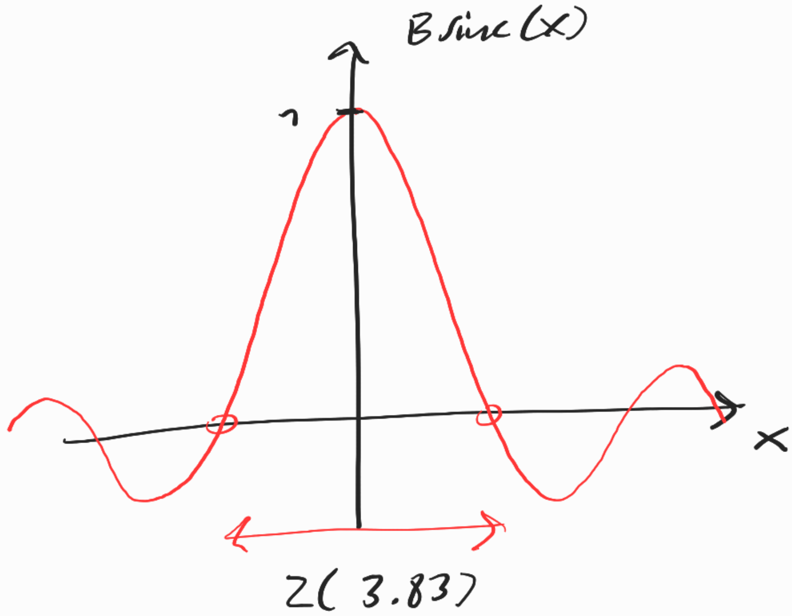
$$\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$$

$$\Rightarrow \underbrace{\frac{d}{dx}\left(x J_1(x)\right)}_{= J_1(x) + x J_1'(x)} = x J_0(x) \quad \Big| \quad J_1'(x) = J_0(x) - \frac{J_1(x)}{x}$$

pero: $\lim_{x\to 0} \frac{J_1(x)}{x} = \lim_{x\to 0} \underbrace{J_1'(x)}_{J_0(x) - \frac{J_1(x)}{x}}$:

$$2\lim_{x\to 0} \frac{J_1(x)}{x} = \lim_{x\to 0} J_0(x) = 1$$

$$\boxed{\lim_{x\to 0} \frac{2J_1(x)}{x} = 1}$$



. radio de la zona central:

se corresponde con el primer cero (no nulo) de J_1:

$x_1 \simeq 3.83$

$J_1(3.83) \approx 0$: $2\pi\rho R \sim 3.83$: $\boxed{\rho R \sim 0.61}$

Transformada 3D:

$$G(f_x, f_y, f_z) = \iiint g(x,y,z)\, e^{-j2\pi(f_x x + f_y y + f_z z)}\, dx\,dy\,dz$$

muy útil para:
- . deconvolución 3D
- . reconstrucción tomográfica usando el 'central slice theorem'