Alternative thermodynamic cycle for the Stirling machine

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We develop an alternative thermodynamic cycle for the Stirling machine, where the polytropic process plays a central role. Analytical expressions for pressure and temperatures of the working gas are obtained as a function of the volume and the parameter that characterizes the polytropic process. This approach achieves closer agreement with the experimental pressure-volume diagram and can be adapted to any type of Stirling engine.
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I. INTRODUCTION

The year 2016 marked the bicentenary of the submission of a patent by Robert Stirling that described his famous engine.[1,2] A Stirling engine is a mechanical device that operates in a closed regenerative thermodynamic cycle, with cyclic compressions and expansions of the working fluid at different temperatures. The flow of the working fluid is controlled only by the internal volume changes; there are no valves and there is a net conversion of heat into work or vice-versa. This engine can run on any heat source (including solar heating), and if combustion-heated it produces very low levels of harmful emissions. A Stirling-cycle machine can be constructed in a variety of different configurations. For example, the expansion-compression mechanisms can be embodied as turbo-machinery, as a piston-cylinder, or even using acoustic waves. Most commonly, Stirling-cycle machines use a piston-cylinder, in either an α, β, or γ configuration.[3]

At the present time, several researchers are working to improve the basic ideas of the Stirling engine.[4–7] The entire engine is a sophisticated combination of simple ideas. The challenge is to make such devices cheap enough to generate electrical power economically. It seems likely that the biggest role for Stirling engines in the future will be to create electricity for local use.

The adequate theoretical description of the thermodynamic cycle for the Stirling machine is not obvious and in general it is necessary to adopt certain simplifications. Usually, the thermodynamic cycle is modeled by alternating two isothermal and two isometric processes. This model has the virtue to give a simple theoretical base, but the real thermodynamic process is quite different. In this paper, we develop an alternative approach, where the polytropic process plays a central role in the cycle. We provide an analytical expression for the pressure of the fluid as a function of its volume and the parameter that characterizes the polytropic process.

The paper is organized as follows. In Sec. II, we treat some aspects of the usual thermodynamic cycle for the Stirling engine, and in Sec. III, we introduce the alternative cycle. In Sec. IV, we introduce the kinematics of the engine in order to complete the model. In the final section, we present our main conclusions.

II. USUAL STIRLING CYCLE

The usual Stirling cycle consists of four reversible processes involving pressure and volume changes. We show these processes, plotted in dashed lines in the pressure-volume diagram of Fig. 1. It is an ideal thermodynamic cycle made up of two isothermal and two isometric regenerative processes. The relation between the movements of the pistons and the processes of the cycle is explained in basic thermodynamics textbooks.[8] The net result of the Stirling cycle is the absorption of heat Q_H at the high temperature T_H, the rejection of heat Q_L (<0) at the low temperature T_L, and the delivery of work $W = Q_H + Q_L$, with no net heat transfer resulting from the two constant-volume processes. This usual cycle is very useful for understanding some qualitative aspects of the Stirling machine; however, it is only a course approximation to the real experimental cycle. The thermal efficiencies of the best Stirling engines can be as high as those of a Diesel engine,[9] and theoretically, they have the Carnot efficiency.

Practical Stirling-cycle machines differ from the usual ideal cycle in several important aspects:[3] (i) the regenerator and heat-exchangers in practical Stirling-cycle machines have nonzero volume, which means that the working gas is never completely in either the hot or cold zone of the machine and therefore never at a uniform temperature; (ii) the motion of the pistons is usually semi-sinusoidal rather than discontinuous, leading to non-optimal manipulation of the working gas; (iii) the expansion and compression processes are better approximated as polytropic rather than isothermal, which allows for pressure and temperature fluctuations in the working gas and leads to adiabatic and transient heat transfer losses; (iv) fluid friction losses occur during gas displacement, particularly due to flow through the regenerator; (v) other factors such as heat conduction between the hot and cold zones of the machine, seal leakage and friction, and friction in kinematic mechanisms all cause real Stirling-cycle machines to differ from ideal behavior. The above factors tend to reduce the performance of real machines.

III. ALTERNATIVE STIRLING CYCLE

As an alternative to the cycle described in Sec. II, the cycle proposed in this section consists of a polytropic process for the working gas. Textbooks usually define a polytropic process for an ideal gas through the relation

$$PV^{\beta} = \text{constant}, \tag{1}$$

where P is the pressure of the gas, V its volume, and β the polytropic index. Let us briefly review the main characteristics of such a process. The quasistatic process is carried out in such a way that the specific heat c remains constant.[11–14]

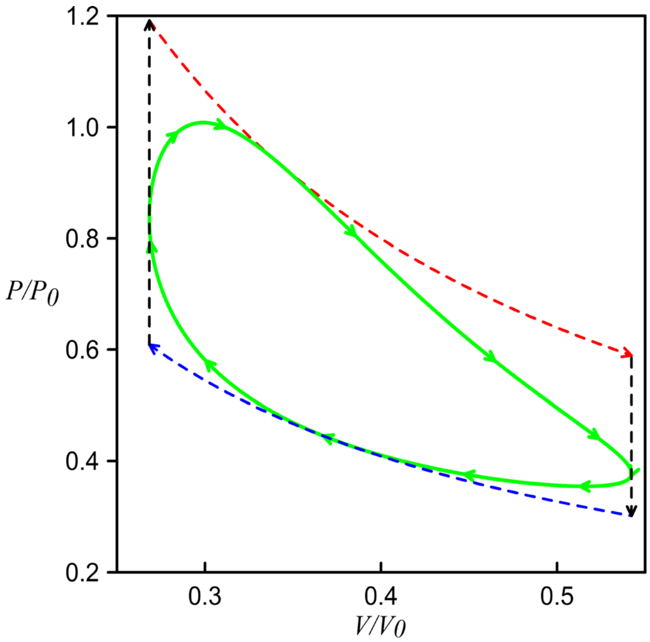

Fig. 1. Dimensionless pressure-volume diagram. The alternative Stirling cycle is presented as the solid curve (green online) compared to the usual Stirling cycle shown as the dashed cycle. The arrows indicate the evolution with increasing θ. The parameters values are $\phi = 5.2\pi$, $\alpha = 10$, $\beta = 1.33$, $\varepsilon = 0.1$, and $z = 0.1$.

Therefore, the relation between heat and temperature is given by

$$dQ = c\,N\,dT, \tag{2}$$

where dQ is the heat absorbed by the gas, T is the absolute temperature, and N is the number of moles. The value of c determines the relation between pressure and volume for the process. A process for which the pressure or the volume is kept constant is, of course, polytropic with specific heat c_P or c_V, respectively. An adiabatic process is a polytropic process with $c = 0$. At the other extreme, isothermal evolution can be thought of as a polytropic process with infinite specific heat. The connection between the polytropic index and the specific heat is given by

$$\beta = \frac{c_P - c}{c_V - c}, \tag{3}$$

$$c = c_V \frac{\gamma - \beta}{1 - \beta}, \tag{4}$$

where $\gamma = c_P/c_V$. Note that if $\beta \in [1, \gamma]$, then $c < 0$.

Next, we shall develop an alternative approach for the Stirling cycle using a simple model for its engine. Nowadays, it easy to find videos on the Internet that present the construction and operation of several toy models of the Stirling engine. We choose one of the simpler of these models[10]—one that works well, is not expensive, and can be built by pre-university students—to develop our ideas about its thermodynamics. This device is schematically shown in Fig. 2. The pistons are connected to the camshaft with an incorporated flywheel. As the shaft rotates, the pistons move with a constant phase difference. The two cylinders are filled with a fixed mass of air, which is recycled from one cylinder to the other. One of the cylinders is kept in contact with the high-temperature reservoir, while the

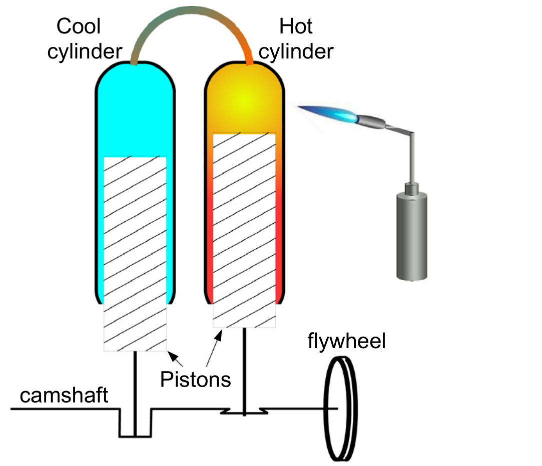

Fig. 2. Simple model of the Stirling engine; it uses two disposable glass syringes as a piston-cylinder system (see Ref. 10).

other is in contact with the low-temperature reservoir. The connection between the cylinders is through a small tube that may have a sponge device a called regenerator.

The model used in this paper has no regenerator and we assume some simplifications for the working gas: (i) the gas has a uniform temperature in each cylinder, T_2 for the hot cylinder and T_1 for the cool cylinder; (ii) the mass of gas inside the tube that connects the cylinders is negligible; (iii) due to the connection between the cold and hot cylinders, the gas always has a uniform pressure P; and (iv) the gas is considered to behave as a classical ideal gas.

In this context, the relation between the internal energy E of the gas inside the Stirling machine and the temperature is given by

$$E = c_V (N_1 T_1 + N_2 T_2), \tag{5}$$

where c_V is constant and N_1 and N_2 are the number of moles of the gas inside of the cool and hot cylinders, respectively. From Eq. (5), the energy change is calculated to be

$$dE = c_V (N_1\,dT_1 + N_2\,dT_2) + c_V\,(T_1 - T_2)\,dN_1, \tag{6}$$

where we have used the conservation of the total number of moles $N = N_1 + N_2$. The first term on the right-hand side of Eq. (6) gives the change of the mean kinetic energy of the molecules with fixed mole numbers. The second term corresponds to the energy change due to the mass redistribution in the cylinders, with fixed temperatures.

In order to analyze the evolution of the system, we compare Eq. (6) to the first law of thermodynamics

$$dE = dQ - dW. \tag{7}$$

The infinitesimal work is given by

$$dW = P\,dV, \tag{8}$$

where dV is the infinitesimal change of the total gas volume V.

The total absorbed (or emitted) heat dQ is given by

$$dQ = c\,(N_1\,dT_1 + N_2\,dT_2) + dQ', \tag{9}$$

where the first term on the right-hand side represents the heat associated with the polytropic process in a bipartite system with two different temperatures, and the second term dQ' is the heat associated with the gas transferred from one cylinder to the other. Therefore, Eq. (7) may be rewritten as

$$dE = -P\,dV + c\,(N_1\,dT_1 + N_2\,dT_2) + dQ'. \tag{10}$$

Comparing Eq. (6) with Eq. (10) and taking into account that the redistribution of the molecules in the cylinders does not produce any work in this system it is clear that

$$dQ' = c_V\,(T_1 - T_2)\,dN_1, \tag{11}$$

and so

$$P\,dV = (c - c_V)\,(N_1\,dT_1 + N_2\,dT_2). \tag{12}$$

On the other hand, using only the equation of state of the ideal gas, we obtain the following expressions for the gas in the cylinders

$$\frac{N_1}{N} = \frac{V_1\,T_2}{V_1\,T_2 + V_2\,T_1}, \tag{13}$$

$$\frac{N_2}{N} = \frac{V_2\,T_1}{V_1\,T_2 + V_2\,T_1}, \tag{14}$$

$$\frac{P}{P_0} = \left(\frac{T_1\,T_2}{T_{10}\,T_{20}}\right)\left(\frac{V_{10}\,T_{20} + V_{20}\,T_{10}}{V_1\,T_2 + V_2\,T_1}\right), \tag{15}$$

where V_1 and V_2 are the volumes of gas in the cylinders, $\{V_{10}, T_{10}\}$, $\{V_{20}, T_{20}\}$, and P_0 are the initial conditions. (Note that $V_1 + V_2 = V$).

Using the previous results, we can rewrite Eq. (12) as

$$V_1\frac{dT_1}{T_1} + V_2\frac{dT_2}{T_2} = (1-\beta)(dV_1 + dV_2). \tag{16}$$

Equation (16) is clearly symmetric with respect to an interchange of the indices 1 and 2. Then, its solution (T_1 and T_2) must be symmetric in the arguments V_1 and V_2 and their functional forms must be essentially the same. Introducing the partial differentiation of the temperatures, Eq. (16) can also be written in the symmetrical form

$$\left(\frac{V_1}{T_1}\frac{\partial T_1}{\partial V_1} + \frac{V_2}{T_2}\frac{\partial T_2}{\partial V_1} - 1 + \beta\right)dV_1 + \left(\frac{V_1}{T_1}\frac{\partial T_1}{\partial V_2} + \frac{V_2}{T_2}\frac{\partial T_2}{\partial V_2} - 1 + \beta\right)dV_2 = 0. \tag{17}$$

In this equation, the variations dV_1 and dV_2 are completely arbitrary. Accordingly, the only way to satisfy this condition is that both expressions in parentheses in Eq. (17) must vanish. Then, it is easy to show that

$$T_1 = T_{10}\left(\frac{V_0}{V}\right)^{\beta-1}, \tag{18}$$

$$T_2 = T_{20}\left(\frac{V_0}{V}\right)^{\beta-1}, \tag{19}$$

satisfy both requisites, where V_0 is the total initial volume.

From Eqs. (15), (18), and (19), the pressure of the working gas is expressed as

$$P = P_0\left(\frac{\alpha\,V_{10} + V_{20}}{\alpha\,V_1 + V_2}\right)\left(\frac{V_0}{V}\right)^{\beta-1}, \tag{20}$$

where

$$\alpha \equiv T_{20}/T_{10}. \tag{21}$$

Equations (18)–(20) prove explicitly that the system undergoes a polytropic process. Using these results and Eqs. (9) and (11), we obtain a differential equation for the heat absorbed in the process

$$dQ = \frac{P}{\gamma - 1}\left[(\gamma-\beta)dV + (1-\alpha)\frac{V_2 dV_1 - V_1 dV_2}{\alpha V_1 + V_2}\right]. \tag{22}$$

The thermodynamic properties of the Stirling engine are determined by Eqs. (18)–(20) and (22). These equations are the principal theoretical results of this paper. In Sec. IV, we derive some properties of the Stirling engine using the above results.

IV. NUMERICAL IMPLEMENTATION

In order to complete the description of the Stirling engine, we develop a simple model for the interaction between the pistons of the engine and the working gas. With this model, it is possible to implement numerical calculations and to obtain some characteristic results of the Stirling engine.

The operation of the Stirling engine can be described essentially in the following way. The working gas follows the thermodynamic cycle interacting with the pistons, which are connected with the camshaft and perform periodic movements associated with the flywheel rotation. The inertia of the flywheel collaborates to sustain the continuity of the movement.

Figure 3 shows the details of the connection between a piston and the camshaft through the rod of length R. The cam has a rotating radius r and the piston has a cross-sectional area a. From this figure, the functional relation between the gas volume and the camshaft angle θ, for both cylinders, is obtained

$$\frac{V_1}{R\,a} = 1 + \varepsilon + z(1 - \cos\theta) - \sqrt{1 - z^2\sin^2\theta}, \tag{23}$$

$$\frac{V_2}{R\,a} = 1 + \varepsilon + z[1 - \cos(\theta+\phi)] - \sqrt{1 - z^2\sin^2(\theta+\phi)}, \tag{24}$$

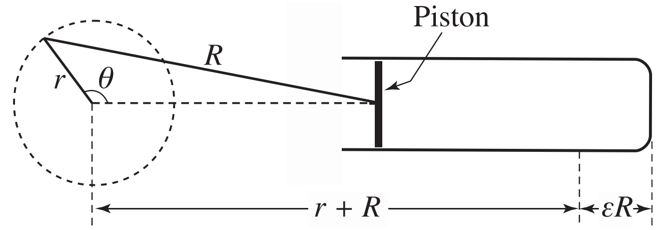

Fig. 3. Constraint scheme between the volume of gas in the cylinder and the camshaft angle θ. The circle represents the cam rotation. The piston is considered to have a negligible thickness.

where $z = r/R$, ε determines the minimum volume of gas in each cylinder during the cycle (we assume the same value for both cylinders), and ϕ is the phase difference between the cams. In what follows, we obtain the initial conditions using $\theta_0 = 0$.

The pressure-volume diagram for the alternative cycle is shown in Fig. 1, which is obtained using Eqs. (20), (23), and (24) and additional numerical calculations. We obtain a smooth continuous curve closer to the experimental behavior. The usual cycle is also shown for comparison with the same highest and lowest temperatures as the alternative cycle. The area inside the solid curve represents the total work of the cycle

$$W = \oint P\,dV = \int_0^{2\pi} P\,\frac{dV}{d\theta}\,d\theta, \tag{25}$$

this is the work available for overcoming mechanical friction losses and for providing useful power to the engine crankshaft.

The alternative cycle does not have four sharply defined processes of the ideal cycle. This approach allows us to model adequately the following facts: (i) the compression and expansion processes do not take place wholly in one or other of the cylinders; (ii) the motion of the pistons is continuous rather than discontinuous; and (iii) the heat exchange between gas and environment is best modeled by a polytropic process rather than an isothermal-isometric sequence.

The pressure-volume diagrams for the hot and cool cylinders are shown in Fig. 4. They are obtained numerically using the same data of Fig. 1. As in the previous figure, the areas of the curves represent the work of each cycle. It is seen that their orientations are opposite, so the available work for the engine is proportional to the difference between these areas.

In Fig. 5, the work per cycle of the Stirling machine is presented as a function of ϕ. The work has a maximum value that depends on the initial conditions and the parameters, as

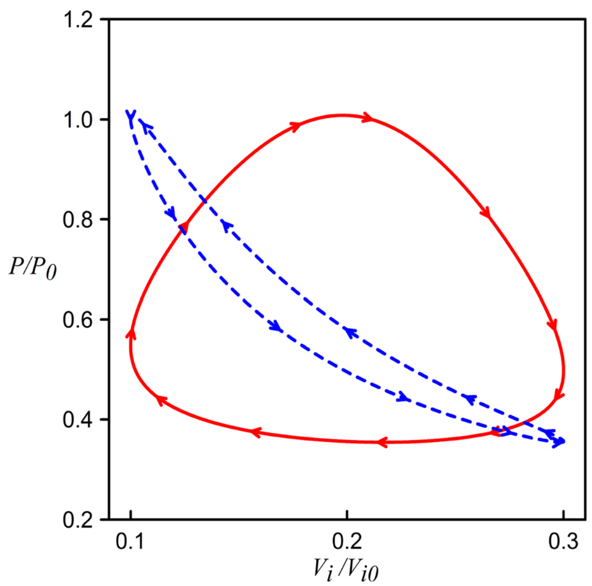

Fig. 4. Dimensionless pressure-volume diagram for the hot gas (solid curve, red online) and the cool gas (dashed curve, blue online). For the hot (cool) gas, the abscissa axis is V_2/V_{20} (V_1/V_{10}). The arrows indicate the evolution with increasing θ. The values of the parameters are the same as in Fig. 1.

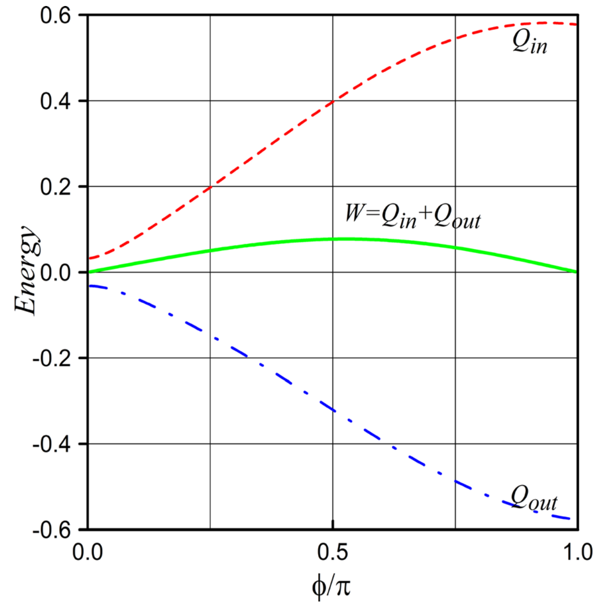

Fig. 5. The energy per cycle as a function of the phase difference ϕ between the cams. The solid curve (green online) is the work, the dashed curve (red online) the heat absorbed, and the dot-dashed curve (blue online) the heat rejected. The values of the parameters are the same as in Fig. 1. The maximum work corresponds to $\phi = 0.52\pi$.

given in the caption. It is known empirically that to obtain the maximum power of the machine, the phase difference between the two cams must be near $\pi/2$. The present model gives us a method to numerically determine this precise angle.

Let us call $Q_{\rm in}$ and $Q_{\rm out}$, respectively, the heat absorbed and rejected by the gas in the cycle. These quantities are calculated numerically using Eq. (22) with the convention $Q_{\rm in} > 0$ and $Q_{\rm out} < 0$, that is,

$$Q_{\rm in} = \int_{\mathcal{C}_{\rm in}} \frac{dQ}{d\theta}\,d\theta, \tag{26}$$

$$Q_{\rm out} = \int_{\mathcal{C}_{\rm out}} \frac{dQ}{d\theta}\,d\theta, \tag{27}$$

where $\mathcal{C}_{\rm in}$ and $\mathcal{C}_{\rm out}$ refer to the paths where $dQ/d\theta > 0$ and $dQ/d\theta < 0$, respectively. Because the internal energy change in a complete cycle vanishes, the total work is $W = Q_{\rm in} + Q_{\rm out}$ as shown in Fig. 5.

The machine efficiency is defined as $W/Q_{\rm in}$, or equivalently

$$\eta = 1 + Q_{\rm out}/Q_{\rm in}. \tag{28}$$

The efficiency as a function of α is shown in Fig. 6. It is easy to prove from Eqs. (20) and (22) that if $\alpha = 1$ then $\eta = 0$; this means, as expected, that without a difference of temperature the machine does not work. When α grows η also grows, however, the efficiency is asymptotically bounded by some value below 0.2. Additionally, we have numerically checked that the efficiency can be improved by reducing the value of ε.

Figure 7 shows the efficiency as a function of the phase difference between the cams. Note that the maximum efficiency does not coincide with the maximum work. This fact is highlighted in Fig. 8, which shows directly the dependence between work and efficiency with varying ϕ.

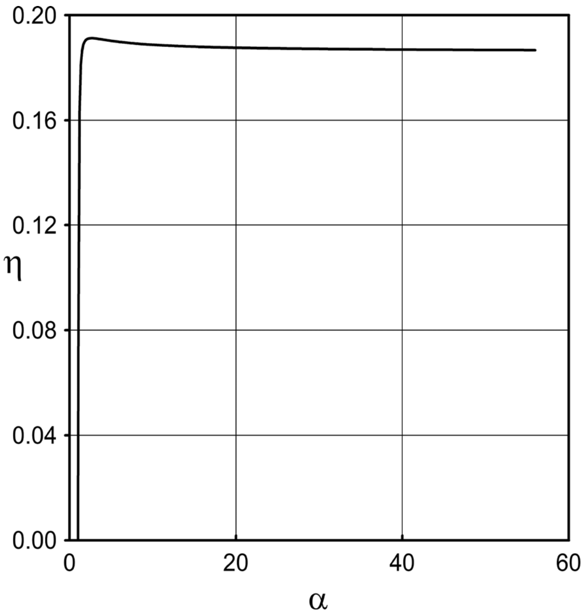

Fig. 6. The efficiency as a function of the parameter α, with $\phi = 0.52\pi$, $\beta = 1.33$, $\varepsilon = 0.1$, and $z = 0.1$.

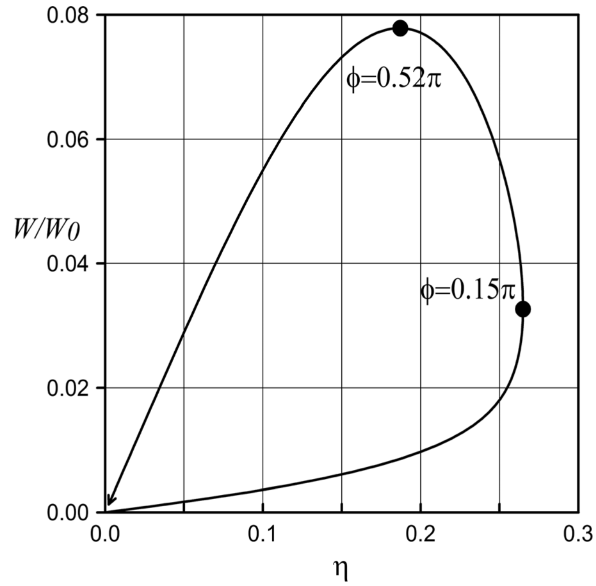

Fig. 8. Dimensionless work as a function of the efficiency η, with the angle ϕ varying between 0 and π. The arrow indicates the direction of increasing ϕ. The parameters are $W_0 = P_0V_0$, $\alpha = 10$, $\beta = 1.33$, $\varepsilon = 0.1$, and $z = 0.1$.

Figure 9 shows the behavior of the work and the efficiency when the parameter β varies. Here, we point out that in Ref. 14 an analytical expression for the polytropic index of air was obtained showing that β depends only on the thermodynamic initial conditions.

Finally, it is interesting to note that our results given by Eqs. (18)–(20) for the isothermal case $\beta = 1$ ($c = -\infty$) coincide with the well-known Schmidt solution,[15–17] published in the year 1871.

V. CONCLUSIONS

This paper develops an alternative theoretical approach to the usual Stirling thermodynamic cycle. This alternative cycle is obtained by applying the first law of thermodynamics to the working fluid inside the Stirling engine. The main characteristic of this approach is the introduction of a polytropic process as a way to represent the exchange of heat with the environment.

We have obtained analytical expressions for the pressure, temperatures, work, and heat for the gas inside the engine. We find that the theoretical pressure-volume diagram shows a qualitative agreement with the experimental diagram. With the aim to complete the description of the Stirling engine, we develop a simple model for the interaction between the pistons of the engine and the working gas and we study the power and the efficiency of the Stirling engine as a function of: (i) the phase difference ϕ between the cams of the engine

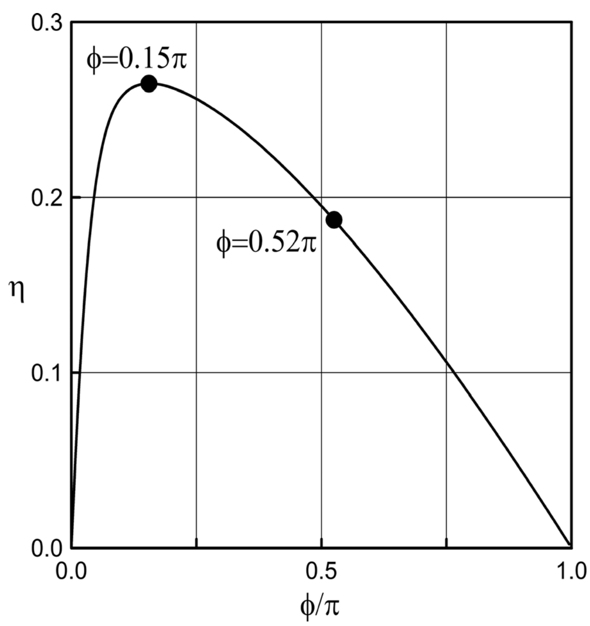

Fig. 7. The efficiency as a function of ϕ, with $\alpha = 10$, $\beta = 1.33$, $\varepsilon = 0.1$, and $z = 0.1$. The dots in the curve indicate the values of ϕ for the maximum efficiency and maximum work.

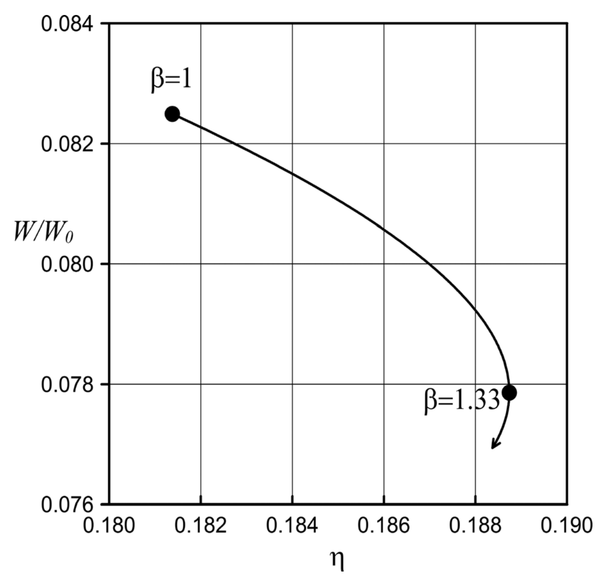

Fig. 9. Dimensionless work as a function of the efficiency η, with $W_0 = P_0V_0$, $\alpha = 10$, $\varepsilon = 0.1$, $\phi = 0.52\pi$, and $z = 0.1$. The polytropic index varies from $\beta = 1.0$ to $\beta = 1.4$. The arrow indicates the direction of increasing β.

crankshaft; (ii) the ratio of the temperatures of the heat sources, α; and (iii) the type of polytropic process, β. We emphasize that the Schmidt analysis for the Stirling machine is a particular case of the solution presented in this paper for $\beta = 1$.

In summary, the theoretical approach proposed in this paper describes the thermodynamics of the Stirling engine in a simple, precise, and natural way that can be adapted to any variant of this engine.

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Double Toepler-Holtz Machine

This four-disk Toepler-Holtz electrostatic machine is on display at the Monroe Moosenick Medical and Science Museum at Transylvania University in Lexington, Kentucky. The basic machine of this type has one stator and rotator. In this example, listed in the 1887 catalogue of James W. Queen of Philadelphia, there are two stators and each provides reference charges to rotators on either side. It cost \$125; the largest model, with plates a yard across, cost \$475, a huge investment at the time. (Picture and Notes by Thomas B. Greenslade, Jr., Kenyon College)