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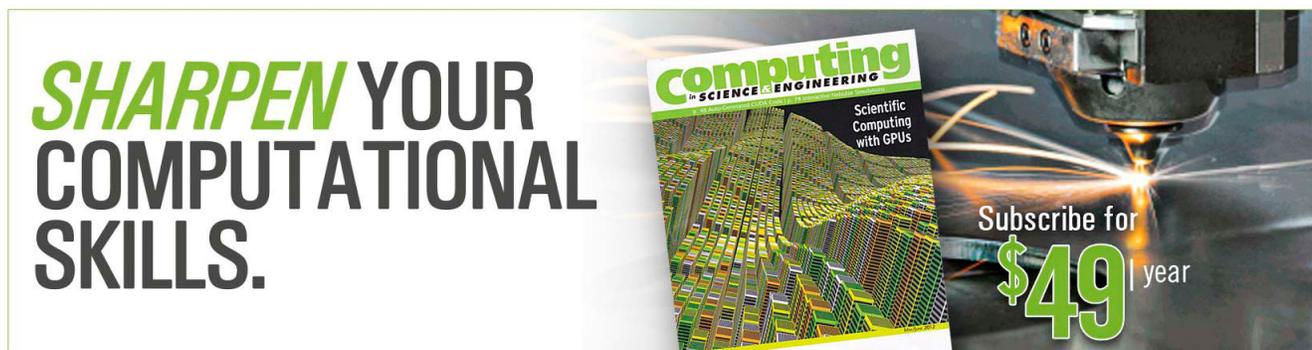
Air expansion in a water rocket

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We study the thermodynamics of a water rocket in the thrust phase, taking into account the expansion of the air with water vapor, vapor condensation, and the corresponding latent heat. We set up a simple experimental device with a stationary bottle and verify that the gas expansion in the bottle is well approximated by a polytropic process PV^{β} = constant, where the parameter β depends on the initial conditions. We find an analytical expression for β that depends only on the thermodynamic initial conditions and is in good agreement with the experimental results. © *2013 American Association of Physics Teachers.*
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I. INTRODUCTION

The water rocket is a popular toy that is already several decades old.[1] An inexpensive version can be built with a plastic bottle filled with water and pressurized air obtained from a bicycle pump. When the bottle is opened, the internal air pressure pushes the water out and the bottle recoils with a corresponding change in momentum. The water ejection gives the bottle the thrust that allows it to move for several meters.[2]

The launch of water rockets is used in our undergraduate laboratory to illustrate several physics concepts: Newton laws, conservation of momentum, the work-energy theorem, the Bernoulli equation, the mass conservation equation, and ideal gas expansion. On this last point, it is necessary to clarify whether the expansion of the gas in the rocket is an adiabatic or an isothermal process.

The theoretical description of the water rocket is not trivial and, in general, it is necessary to adopt certain simplifications. In particular, the rocket's only source of energy, the air expansion, is in general modeled[2–5] as an isothermal or an adiabatic process involving dry air. These theoretical approaches give qualitatively reasonable predictions, but they are quantitatively incorrect, showing that some hypothesis of the model is inadequate. In particular, observing the launch of the water rocket, we see that when the ejection of the water concludes, an additional ejection of fog follows. Therefore, it is clear that the expansion occurs for a mixture of dry air, water vapor, and condensed water (fog). In Ref. 6, the author pursues this point and builds a pressure-volume relation assuming that the total entropy of the mixture is conserved during the expansion; he additionally shows that the solution of the pressure-volume equation can be approximated by a polytropic process.

We have developed an experimental device in order to study the nature of the water-rocket gas expansion. We also present a brief theoretical development that provides an analytical expression for the parameter that characterizes the polytropic processes.

The paper is organized as follows. In Sec. II, we treat some aspects of polytropic processes, and in Sec. III, we study the exchange of heat during condensation in a polytropic process. A theoretical expression for the polytropic exponent of the system is obtained in Sec. IV. Our experimental device and results are presented in Sec. V; this is followed by Sec. VI.

II. CHARACTERIZATION OF POLYTROPIC PROCESSES

We present here a brief review of the main characteristics of polytropic processes for an ideal gas. A polytropic process is a quasistatic process carried out in such a way that the molar specific heat c remains constant.[7–9] Therefore, the heat exchange when the temperature change is

$$dQ = Nc\, dT, \tag{1}$$

where dQ is the heat absorbed by the system, dT is the temperature change, and N is the number of moles. Pressure and volume can change arbitrarily during the process, and the value of c will depend on how they change. A process in which the pressure or the volume is kept constant is of course polytropic with specific heat c_P or c_V, respectively. In an adiabatic process, there is no heat exchange between the system and the environment, so it is a polytropic process with $c = 0$. At the other extreme, isothermal evolution is characterized by a constant temperature and therefore it can be thought of as a polytropic process with infinite specific heat.

Starting from the first law of thermodynamics

$$dU = dQ - P\, dV, \tag{2}$$

where P is the pressure, V is the volume, and

$$dU = c_V N dT \tag{3}$$

is the change in the internal energy of an ideal gas, together with Eq. (1), we obtain a differential equation for the polytropic process

$$(c_V - c)N\, dT + P\, dV = 0. \tag{4}$$

Using Eq. (4) and the ideal gas equation of state, one can show that the polytropic process satisfies

$$PV^{\beta} = \text{constant}, \tag{5}$$

where

$$\beta = \frac{c_P - c}{c_V - c}. \tag{6}$$

Normally thermodynamics textbooks define a polytropic process through Eq. (5) (for example, see Ref. 10); however, we have started with Eq. (1), which gives a direct link between heat and temperature changes. In particular, for an adiabatic process $c = 0$ and $\beta = \gamma$, where

$$\gamma \equiv \frac{c_P}{c_V} \tag{7}$$

($\gamma = 1.4$ for dry air at room temperature). For an isothermal process $c = \infty$ and $\beta = 1$; for an isobaric process $c = c_P$ and $\beta = 0$; and for an isovolumic process $c = c_V$ and $\beta = \infty$.

Solving Eq. (6) for the specific heat gives

$$c = \frac{c_P - \beta c_V}{1 - \beta}. \tag{8}$$

It is well known that a system in stable thermodynamic equilibrium must have $c_V > 0$, otherwise the second law of thermodynamics is contradicted.[13,14] However, the value of c depends on the process and in principle it has no restriction.[15] For polytropic processes, it is seen from Eq. (8) that if $1 < \beta < \gamma$ then $c < 0$. Common examples of such a case in real life are the expansion process in an internal combustion engine and the compression process in a vapor compression refrigerator.

In the experimental system treated in this paper, $c < 0$ means that during the gas expansion phase heat is absorbed ($dQ > 0$) and yet the gas temperature decreases ($dT < 0$); this occurs because the gas performs an amount of work larger than the absorbed heat.

Finally, the total heat Q exchanged between the ideal gas and the environment can be calculated by integrating Eq. (1) to get

$$Q = cN(T - T_0), \tag{9}$$

where T is the gas temperature and T_0 is its initial temperature.

III. HEAT EXCHANGE AND VAPOR CONDENSATION

We now turn to our experimental situation. In the following, we shall use the term "gas" to refer to the mixture of dry air and water vapor. As the water is expelled from the bottle, the gas expands and cools. During this process, the relative humidity increases up to saturation and some of the vapor condenses. The quantity of condensed water vapor at the end of the experiment will depend on the initial relative humidity in the gas and the initial and final temperatures. We want to estimate the mass and the heat involved in this condensation.

The mass of water vapor m_v can be calculated using[12]

$$m_v = \frac{P_v V}{R_v T}, \tag{10}$$

where P_v is the partial pressure of the vapor at T and $R_v = 0.4615$ kJ/(kg K) is the ideal gas constant for the vapor. The vapor partial pressure is determined by the equation $P_v = \varepsilon e_w$, where $\varepsilon \in [0, 1]$ is the (fractional) relative humidity and e_w is the saturation pressure that depends only on the temperature and can be obtained from a vapor table. In our experiment, the pressurized air is generated by the

compressor at a higher temperature than the environment and its cooling results in the saturation of the air with vapor inside the system. Thus, we assume that we are working with $\varepsilon = 1$ throughout the experiment and therefore $P_v = e_w$. This assumption implies the maximization of the energy delivered by the vapor-liquid phase transition. In our case, the initial temperature is $T_0 = 292$ K and the partial pressure is $P_{v_0} = 2.3$ kPa (this last value was taken from a vapor table).

In order to get a fuller understanding of the heat exchange due to vapor condensation, we start with the Clapeyron-Clausius equation[11] for the vapor-liquid phase transition

$$\frac{de_w}{dT} = \frac{L_v}{R_v} \frac{e_w}{T^2}, \tag{11}$$

where L_v is the latent heat, which for our purposes may be considered constant at $L_v = 2500$ kJ/kg. The solution of Eq. (11) gives the partial pressure $P_v = \varepsilon e_w$ as a function of the temperature T to be

$$P_v = P_{v_0} \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right], \tag{12}$$

where the integration constant has been adjusted so that $P_v(T_0) = P_{v_0}$.

The condensation produces a change in the mass of vapor in the gas. The mass of condensed vapor can be calculated using Eq. (10) twice, that is,

$$M = \frac{P_{v_0} V_0}{R_v T_0} - \frac{P_v V}{R_v T}. \tag{13}$$

Using the fact that a polytropic process can also characterized by the equation $PT^{\beta/(1-\beta)} = $ constant, we find

$$\frac{V}{T} = \frac{NR}{P} = \frac{NR}{P_0}\left(\frac{T_0}{T}\right)^{\beta/(\beta-1)}, \tag{14}$$

where R is the ideal gas constant. To obtain the latent heat of condensation

$$Q = ML_v, \tag{15}$$

we substitute Eqs. (12) and (14) into Eq. (13) to obtain

$$Q = \frac{NRP_{v_0}L_v}{R_v P_0}\left\{1 - \left(\frac{T_0}{T}\right)^{\beta/(\beta-1)} \exp\left[\frac{L_v}{T_0 R_v}\left(1 - \frac{T_0}{T}\right)\right]\right\}. \tag{16}$$

One may wonder if there are other types of heat exchange between the system and the environment. As will be seen, our experiment lasts for less than one second so it is reasonable to assume that the heat exchanged through the walls is not important; the energy delivered by the vapor-liquid phase transition remains the sole source of heat for the system.

It is important to point out that, due to the water vapor condensation, the number of moles N in the gas is not constant. However, Eqs. (9) and (16) were obtained using the ideal gas law for air and water vapor, assuming that N and N_v (the number of moles of water vapor) are constants. Therefore, these equations can be used only if the change in the number of moles is negligible during condensation. The

relative number of moles condensed $\delta N_v/N_v$ can be estimated using Eqs. (10) and (13) to be

$$\frac{\delta N_v}{N_v} = \frac{M}{m_{v_0}} = 1 - \left(\frac{T_0}{T}\right)^{\beta/(\beta-1)} \exp\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right], \tag{17}$$

where m_{v_0} is the initial water vapor mass. If we estimate the final temperature as $T = T_0 - (10\,\mathrm{K})$, then Eq. (17) gives $\delta N_v/N_v \sim 0.6$. The total relative variation of moles in the gas $\delta N_v/N$ is then

$$\frac{\delta N_v}{N} = \frac{\delta N_v}{N_v}\frac{N_v}{N} = \frac{P_{v_0}}{P_0}\frac{\delta N_v}{N_v} \sim 10^{-3}, \tag{18}$$

where we have used the ideal gas equation both for the gas and for the water vapor. Thus, $\delta N_v/N$ is negligible while but $\delta N_v/N_v$ is not. In the derivation of Eq. (16), we have used Eq. (10), which is correct only if $\delta N_v/N_v \sim 0$; thus, Eq. (16) can only be correct in the limit $T \to T_0$. Therefore, we must replace Eq. (16) with the exact differential equation

$$dQ = cNdT, \tag{19}$$

where

$$c = -\frac{L_v}{T_0 R_v}\frac{P_{v_0}R}{P_0}\left(\frac{\beta}{1-\beta} + \frac{L_v}{R_v T_0}\right). \tag{20}$$

Equation (19) is obtained by replacing $T = T_0 + dT$ in Eq. (16) and taking the terms to first order in dT.

IV. CALCULATION OF THE POLYTROPIC EXPONENT β

In this section, we calculate the exponent β using the previous results. Equation (8) gives the general expression for the polytropic specific heat c for an ideal gas. On the other hand, Eq. (20) relates the specific heat c to the initial conditions and the parameters of the system. Equating the right-hand sides of Eqs. (8) and (20), we find

$$\frac{c_P/R - \beta c_V/R}{1-\beta} = -\frac{L_v}{T_0 R_v}\frac{P_{v_0}}{P_0}\left(\frac{\beta}{1-\beta} + \frac{L_v}{R_v T_0}\right), \tag{21}$$

which can be solved to give a theoretical expression for the β:

$$\beta = \frac{c_P/R + (L_v/R_v T_0)^2(P_{v_0}/P_0)}{c_V/R + (L_v/R_v T_0)^2(P_{v_0}/P_0) - (L_v/R_v T_0)(P_{v_0}/P_0)}. \tag{22}$$

Equation (22) is the main theoretical result in this paper. Using this equation, we conclude, first, that if $P_{v_0} \sim 0$ or $L_v \sim 0$ then $\beta \sim \gamma = c_P/c_v$; this means that the process is adiabatic for dry air. Second, β depends only on the thermodynamic initial conditions of the gas; other parameters like the water ejection time do not appear. Table I shows the theoretical values of β calculated from Eq. (22), which are labelled β^*, computed for our experimental initial conditions.

The model developed in Ref. 6 also predicts that the exponent β depends mostly on the initial relative humidity and proposes the empirical function

Table I. Experimental and theoretical results for different initial pressures P_0, with P_A the atmosphere pressure, β the experimental value of the polytropic exponent, β^* the theoretical value of the polytropic exponent, c the molar specific heat, R the ideal gas constant, and t the ejection time.

P_0/P_A	β	β^*	c/R	t (s)
4.8	1.22	1.27	−2.0	0.64
4.4	1.21	1.26	−2.2	0.66
4.0	1.20	1.25	−2.6	0.75
3.6	1.19	1.24	−2.7	0.75
3.2	1.17	1.23	−3.5	0.88

$$\beta = 1.15 + (1.4 - 1.15)e^{-36P_{v_0}/P_0}. \tag{23}$$

Figure 1 compares the β values computed from Eqs. (22) and (23) as functions of the initial ratio P_{v_0}/P_0. (This figure also shows the experimental values described in Sec. V.)

V. EXPERIMENTAL RESULTS

In order to elucidate the nature of the expansion of the air in the water rocket, we designed an experimental device to measure the heat exchange between the compressed gas of the water rocket and the environment. The water rocket is fixed in the laboratory frame and the time dependence of the pressure and volume of the expanding gas in the bottle is measured (see Fig. 2).

The bottle is a transparent acrylic cylinder of inner diameter 5.40 cm, wall thickness 0.30 cm, and length 1 m, maintained in a vertical orientation. The upper and lower covers are both made of aluminum. The upper cover has a tire valve and a manometer (WIKA EN 837-1) working from 0 to 4 bars. The lower cover has a conical shape with a slope of 30°, with a 0.5-in. faucet that can fill and evacuate the water in the bottle. The bottle was built in our mechanical workshop. The device is quite simple and the experiment is not dangerous for the range of pressures used in this paper, so it is appropriate for undergraduate students. In a typical trial
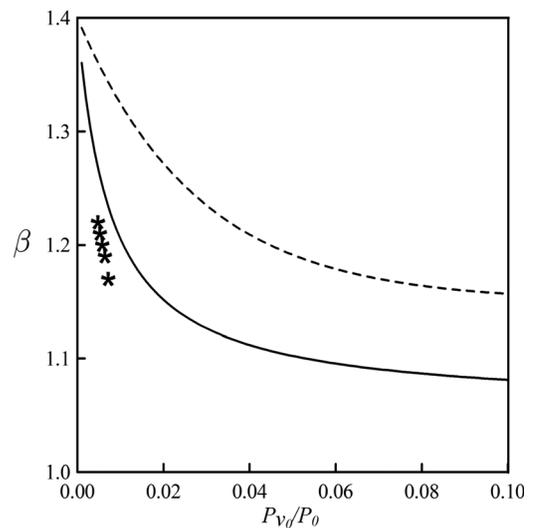
Fig. 1. The polytropic exponent as a function of the initial molar fraction. The star points are our experimental values of β. The solid line is our theoretical calculation given by Eq. (22), while the dashed line is the function given by Eq. (23).
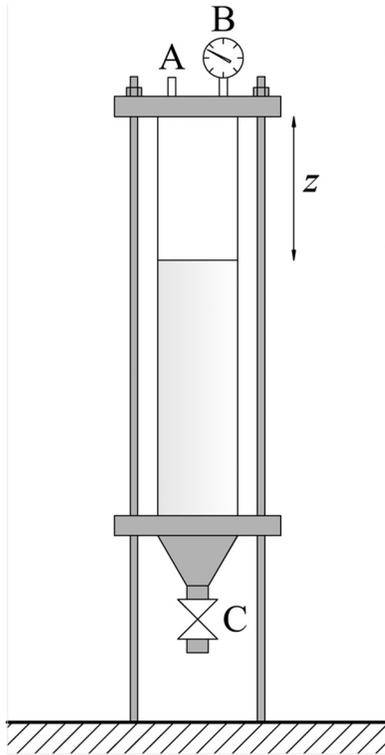
Fig. 2. Schematic diagram of our stationary bottle apparatus showing tire valve (A), pressure manometer (B), faucet (C), and the height z of the gas column situated over the water column.

Fig. 3. The dimensionless pressure as a function of the dimensionless volume of the gas, plotted on a log-log scale. Data from five experiments are shown, together with lines fit to the data. The values of the initial dimensionless pressure are, from top to bottom, 4.8, 4.4, 4.0, 3.6, and 3.2, with P_A the atmospheric pressure and $V_0 = 928.6$ cm^3.

the bottle is filled with about 1400 cm^3 of water and the rest with air. It is placed in a vertical orientation with the faucet closed. The compressed air is introduced through the valve and then the faucet is opened. The level and pressure of the gas are filmed with a PIXELINK PL-B776F camera. The video is recorded at 50 frames per second, with an exposure time of 10 ms per frame. The ejection of water from the bottle lasted between 0.64 and 0.88 s, depending of the initial pressure (see Table I). Finally, the video was processed manually to obtain the pressure and volume of the gas as functions of time.

Figure 3 shows five sets of experimental pressures and volumes. Each set corresponds to a different initial pressure with the same initial volume. For each initial pressure, the experiment was repeated five times and all the data are incorporated into the same figure. The data are fit with a power law $PV^\beta = $ constant, and the values of β (from fitting the slopes in the figure) are given in Table I. These experimental values of the polytropic exponents β are in fair agreement with the theoretical values β^* given by Eq. (22).

VI. DISCUSSION AND CONCLUSION

The typical ejection time of the water in our experiments is 0.7 s (see Table I), while the typical ejection time in the usual water rocket is 0.1 s.[2] The difference in the ejection rates must be analyzed. The main geometrical difference between the bottles is in the nozzles. The standard soda bottle has a nozzle area of 3.46×10^{-4} m^2, while the nozzle area of the stationary bottle was only 40% as large; a smaller nozzle was necessary in order to accommodate the capability of our video camera. Another difference between the experiments is the apparent gravity. In the stationary bottle, the

water is only subjected to its weight, while in a normal water rocket, the water is also subjected to a fictitious force due to the rocket's acceleration; this makes the apparent gravitational field several times larger than the Earth's gravitational field. However, our theoretical calculation shows that β^* does not depend on the water output rate; it depends only on the thermodynamic initial conditions of the gas. Furthermore, β^* agrees reasonably well with our experimental values. We therefore conclude that the thermodynamic process is the same in both the water rocket and in our stationary experiment, given the same thermodynamic initial conditions.

In summary, we have presented an experimental study of the air expansion in the water rocket using a stationary bottle. It is shown that the air expansion follows a polytropic process of the type $PV^\beta = $ constant, where β is an average value with slight dependence on the initial conditions. We have also obtained an analytical expression for the polytropic exponent that agrees reasonably well with the experimental values. This theoretical exponent depends only on the thermodynamic initial conditions of the gas, so it also applies to the air expansion in the usual water rocket. Finally, we conclude that vapor condensation is the main heat source for the gas expansion process in the water rocket.

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Carbon Bisulphide Prism

The index of refraction of carbon bisulphide changes by nearly 4% over the range of visible wavelengths. It thus produces a spectrum with a much larger amount of dispersion than typical glasses. To use it with a spectrometer, it is contained in a hollow prism, like this device listed at $6.00 in the 1929 catalogue of the Chicago Apparatus Company. The pure liquid is supposed to have an "ether-like" odor, but most samples are contaminated with remarkably foul-smelling impurities. The prism is in the apparatus collection of Kenyon College in Gambier, Ohio. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)