Discrete random variables

Bernoulli(p)

$\mathsf{P}(X=1)=p, \quad \mathsf{P}(X=0)=1-p.$
$\mathsf{E}[X]=p, \quad \mathsf{var}(X)=p(1-p), \quad G_X(z)=(1-p)+pz.$

Poisson(λ)

$\mathsf{P}(X=k)=\dfrac{\lambda^k e^{-\lambda}}{k!}, \quad k=0,1,\ldots.$

$\mathsf{E}[X]=\lambda, \quad \mathsf{var}(X)=\lambda, \quad G_X(z)=e^{\lambda(z-1)}.$

geometric$_0$(p)

$\mathsf{P}(X=k)=(1-p)p^k, \quad k=0,1,2,\ldots.$

$\mathsf{E}[X]=\dfrac{p}{1-p}, \quad \mathsf{var}(X)=\dfrac{p}{(1-p)^2}, \quad G_X(z)=\dfrac{1-p}{1-pz}.$

geometric$_1$(p)

$\mathsf{P}(X=k)=(1-p)p^{k-1}, \quad k=1,2,3,\ldots.$

$\mathsf{E}[X]=\dfrac{1}{1-p}, \quad \mathsf{var}(X)=\dfrac{p}{(1-p)^2}, \quad G_X(z)=\dfrac{(1-p)z}{1-pz}.$

binomial(n,p)

$\mathsf{P}(X=k)=\dbinom{n}{k}p^k(1-p)^{n-k}, \quad k=0,\ldots,n.$

$\mathsf{E}[X]=np, \quad \mathsf{var}(X)=np(1-p), \quad G_X(z)=[(1-p)+pz]^n.$

negative binomial or Pascal(m,p)

$\mathsf{P}(X=k)=\dbinom{k-1}{m-1}(1-p)^m p^{k-m}, \; k=m,m+1,\ldots.$

$\mathsf{E}[X]=\dfrac{m}{1-p}, \quad \mathsf{var}(X)=\dfrac{mp}{(1-p)^2}, \quad G_X(z)=\left[\dfrac{(1-p)z}{1-pz}\right]^m.$

Note that Pascal$(1,p)$ is the same as geometric$_1(p)$.

Note. To obtain the moment generating function from $G_X(z)$, let $z=e^s$. To obtain the characteristic function, let $z=e^{j\nu}$. In other words, $M_X(s)=G_X(e^s)$ and $\varphi_X(\nu)=G_X(e^{j\nu})$.

Fourier transforms

Fourier transform

$$H(f) \;=\; \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt$$

Inversion formula

$$h(t) \;=\; \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}\,df$$

$h(t)$	$H(f)$
$I_{[-T,T]}(t)$	$2T\dfrac{\sin(2\pi Tf)}{2\pi Tf}$
$2W\dfrac{\sin(2\pi Wt)}{2\pi Wt}$	$I_{[-W,W]}(f)$
$(1-\lvert t\rvert/T)I_{[-T,T]}(t)$	$T\left[\dfrac{\sin(\pi Tf)}{\pi Tf}\right]^2$
$W\left[\dfrac{\sin(\pi Wt)}{\pi Wt}\right]^2$	$(1-\lvert f\rvert/W)I_{[-W,W]}(f)$
$e^{-\lambda t}u(t)$	$\dfrac{1}{\lambda+j2\pi f}$
$e^{-\lambda\lvert t\rvert}$	$\dfrac{2\lambda}{\lambda^2+(2\pi f)^2}$
$\dfrac{\lambda}{\lambda^2+t^2}$	$\pi e^{-2\pi\lambda\lvert f\rvert}$
$e^{-(t/\sigma)^2/2}$	$\sqrt{2\pi}\,\sigma\, e^{-\sigma^2(2\pi f)^2/2}$

Note. The indicator function $I_{[a,b]}(t) := 1$ for $a \le t \le b$ and $I_{[a,b]}(t) := 0$ otherwise. In particular, $u(t) := I_{[0,\infty)}(t)$ is the unit step function.

Series formulas

$$\sum_{k=0}^{N-1} z^k \;=\; \frac{1-z^N}{1-z}, \quad z \ne 1 \qquad e^z \;:=\; \sum_{k=0}^{\infty}\frac{z^k}{k!} \qquad (a+b)^n \;=\; \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$

$$\sum_{k=0}^{\infty} z^k \;=\; \frac{1}{1-z}, \quad |z|<1 \qquad \lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n \;=\; e^z \qquad \sum_{k=1}^{n} k \;=\; \frac{n(n+1)}{2}$$

Continuous random variables

uniform$[a,b]$

$f_X(x) = \dfrac{1}{b-a}$ and $F_X(x) = \dfrac{x-a}{b-a}$, $a \le x \le b$.

$\mathsf{E}[X] = \dfrac{a+b}{2}$, $\mathsf{var}(X) = \dfrac{(b-a)^2}{12}$, $M_X(s) = \dfrac{e^{sb}-e^{sa}}{s(b-a)}$.

exponential, exp(λ)

$f_X(x) = \lambda e^{-\lambda x}$ and $F_X(x) = 1 - e^{-\lambda x}$, $x \ge 0$.

$\mathsf{E}[X] = 1/\lambda$, $\mathsf{var}(X) = 1/\lambda^2$, $\mathsf{E}[X^n] = n!/\lambda^n$.

$M_X(s) = \lambda/(\lambda - s)$, $\operatorname{Re} s < \lambda$.

Laplace(λ)

$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}$.

$\mathsf{E}[X] = 0$, $\mathsf{var}(X) = 2/\lambda^2$. $M_X(s) = \lambda^2/(\lambda^2 - s^2)$, $-\lambda < \operatorname{Re} s < \lambda$.

Cauchy(λ)

$f_X(x) = \dfrac{\lambda/\pi}{\lambda^2 + x^2}$, $F_X(x) = \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{x}{\lambda}\right) + \dfrac{1}{2}$.

$\mathsf{E}[X] = \text{undefined}$, $\mathsf{E}[X^2] = \infty$, $\varphi_X(\nu) = e^{-\lambda|\nu|}$.

Odd moments are not defined; even moments are infinite. Since the first moment is not defined, central moments, including the variance, are not defined.

Gaussian or normal, $N(m, \sigma^2)$

$f_X(x) = \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right]$. $F_X(x) = \texttt{normcdf(x,m,sigma)}$.

$\mathsf{E}[X] = m$, $\mathsf{var}(X) = \sigma^2$, $\mathsf{E}[(X-m)^{2n}] = 1\cdot 3 \cdots (2n-3)(2n-1)\sigma^{2n}$,

$M_X(s) = e^{sm + s^2\sigma^2/2}$.

Rayleigh(λ)

$f_X(x) = \dfrac{x}{\lambda^2} e^{-(x/\lambda)^2/2}$ and $F_X(x) = 1 - e^{-(x/\lambda)^2/2}$, $x \ge 0$.

$\mathsf{E}[X] = \lambda\sqrt{\pi/2}$, $\mathsf{E}[X^2] = 2\lambda^2$, $\mathsf{var}(X) = \lambda^2(2 - \pi/2)$.

$\mathsf{E}[X^n] = 2^{n/2}\lambda^n \Gamma(1 + n/2)$.

Note. To obtain the characteristic function from $M_X(s)$, let $s = j\nu$. In other words, $\varphi_X(\nu) = M_X(j\nu)$.

Continuous random variables

gamma(p, λ)

$f_X(x) = \lambda \dfrac{(\lambda x)^{p-1} e^{-\lambda x}}{\Gamma(p)}$, $x > 0$, where $\Gamma(p) := \int_0^\infty x^{p-1} e^{-x}\,dx$, $p > 0$.

Recall that $\Gamma(p) = (p-1)\cdot\Gamma(p-1)$, $p > 1$.

$F_X(x) = \texttt{gamcdf(x,p,1/lambda)}$.

$\mathsf{E}[X^n] = \dfrac{\Gamma(n+p)}{\lambda^n \Gamma(p)}$. $M_X(s) = \left(\dfrac{\lambda}{\lambda - s}\right)^p$, $\operatorname{Re} s < \lambda$.

Note that gamma$(1,\lambda)$ is the same as exp(λ).

Erlang(m,λ) := gamma(m,λ), m = integer

Since $\Gamma(m) = (m-1)!$

$f_X(x) = \lambda \dfrac{(\lambda x)^{m-1} e^{-\lambda x}}{(m-1)!}$ and $F_X(x) = 1 - \displaystyle\sum_{k=0}^{m-1} \dfrac{(\lambda x)^k}{k!} e^{-\lambda x}$, $x > 0$.

Note that Erlang$(1,\lambda)$ is the same as exp(λ).

chi-squared(k) := gamma$(k/2, 1/2)$

If k is an even integer, then chi-squared(k) is the same as Erlang$(k/2, 1/2)$.

Since $\Gamma(1/2) = \sqrt{\pi}$,

for $k = 1$, $f_X(x) = \dfrac{e^{-x/2}}{\sqrt{2\pi x}}$, $x > 0$.

Since $\Gamma\left(\frac{2m+1}{2}\right) = \frac{(2m-1)\cdots 5\cdot 3\cdot 1}{2^m}\sqrt{\pi}$,

for $k = 2m+1$, $f_X(x) = \dfrac{x^{m-1/2} e^{-x/2}}{(2m-1)\cdots 5\cdot 3\cdot 1\sqrt{2\pi}}$, $x > 0$.

$F_X(x) = \texttt{chi2cdf(x,k)}$.

Note that chi-squared(2) is the same as exp$(1/2)$.

Weibull(p,λ)

$f_X(x) = \lambda p x^{p-1} e^{-\lambda x^p}$ and $F_X(x) = 1 - e^{-\lambda x^p}$, $x > 0$.

$\mathsf{E}[X^n] = \dfrac{\Gamma(1 + n/p)}{\lambda^{n/p}}$.

Note that Weibull$(2,\lambda)$ is the same as Rayleigh$\left(1/\sqrt{2\lambda}\right)$ and that Weibull$(1,\lambda)$ is the same as exp(λ).