Almost all authors of college physics books are confused about the proper use of Faraday's Law in circuits with inductors. Giancoli is no exception. Professor **John Belcher** (who has lectured 8.02 many times) has a wonderful Lecture Supplement which sets the record straight. It follows below. I (Walter Lewin) have amended it slightly by adding references to Giancoli (Belcher used a different book which made the same embarrassing mistakes) and by referencing my 8.02 lecture of March 15, 2002. It may help to first read the lecture supplement on non-conservative fields of March 15.

Self-Inductance — Kirchhoff's 2nd Law — Faraday's Law

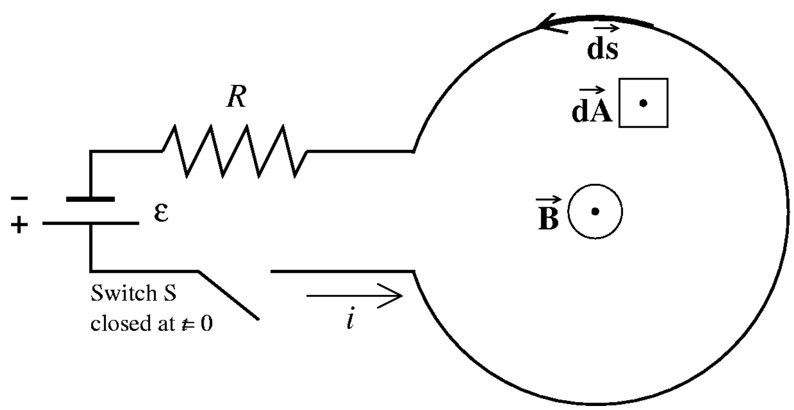
The addition of time-changing magnetic fields to simple circuits means that the closed line integral of the electric field around a circuit is no longer zero. Instead, we have, for any open surface

$$\oint \vec{\mathbf{E}} \cdot \vec{\mathbf{ds}} = -\frac{d}{dt}\int \vec{\mathbf{B}} \cdot \vec{\mathbf{dA}}$$

Any circuit where the current changes with time will have time-changing magnetic fields, and therefore induced electric fields. How do we solve simple circuits taking such effects into account? We discuss here a consistent way to understand the consequences of introducing time-changing magnetic fields into circuit theory — that is, *inductance.*

As soon as we introduce time-changing magnetic fields, the electric potential difference between two points in our circuit is no longer well-defined, because when the line integral of the electric field around a closed loop is no longer zero, the potential difference between points a and b, say, is no longer independent of the path used to get from point a to point b. That is, the electric field is no longer a conservative field, and the electric potential is no longer an appropriate concept ($\vec{\mathbf{E}}$ can no longer be written as the negative gradient of a scalar potential). However, we can still write down in a straight-forward fashion the equation that determines the behavior of a circuit.

To show how to do this, consider the circuit shown in the sketch to the right. We have a battery, a resistor, a switch S that is closed at $t = 0$, and a "one-loop inductor." It will become clear what the consequences of this "inductance" are as we proceed. For $t > 0$, current will flow in the direction shown (from the positive terminal of the battery to the negative, as usual). What is the equation that governs the behavior of our current i for $t > 0$?



To investigate this, apply Faraday's Law to the open surface bounded by our circuit, where we take $\vec{\mathbf{dA}}$ out of the page, and $\vec{\mathbf{ds}}$ right-handed with respect to that choice (counter-clockwise). First, what is the integral of the electric field around this circuit? Well, there is an electric field in the battery, directed from the positive terminal to the negative terminal, and when we go through the battery in the direction of $\vec{\mathbf{ds}}$ that we have chosen, we are moving against that electric field, so that $\vec{\mathbf{E}} \cdot \vec{\mathbf{ds}}$ is negative. Thus the contribution of the battery to our integral is $-\varepsilon$. Then, there is an electric field in the resistor, in the direction of the current, so when we move through the resistor in that direction, $\vec{\mathbf{E}} \cdot \vec{\mathbf{ds}}$ is positive, and that contribution to our integral is $+iR$. What about when we move through our "one-loop inductor"? There is no electric field in this loop if the resistance of the wire making up the loop is zero (this may bother you — if so, see the next section). If the wire has a small resistance $r \ll R$, then there will be an electric field in the wire, and a contribution to the integral of the electric field of $+ir$, which we just lump with the iR term we already have (that is, we redefine R to include both resistances). So, going totally around the closed loop, we have

$$\oint \vec{\mathbf{E}} \cdot \vec{\mathbf{ds}} = -\varepsilon + iR$$

Now, what is the magnetic flux ϕ through our open surface? First of all, we arrange the geometry so that the part of the circuit which includes the battery, the switch, and the resistor makes only a small contribution to ϕ as compared to the (much larger in area) part of the open surface which includes our "one-loop" inductor. Second, we know that ϕ is positive in that part of the surface, because current flowing counter-clockwise will produce a $\vec{\mathbf{B}}$ field out of the paper, which is the same direction we have assumed for $\vec{\mathbf{dA}}$, so $\vec{\mathbf{B}} \cdot \vec{\mathbf{dA}}$ is positive. Note that $\vec{\mathbf{B}}$ is the *self* magnetic field — that is the magnetic field produced by the current flowing in the circuit, and not by any external currents.

We also know that at any point in space, $\vec{\mathbf{B}}$ is proportional to the current i, since it is computed from the Biot-Savart Law, to wit,

$$\vec{\mathbf{B}} = i \oint \frac{\mu_0}{4\pi} \frac{\vec{\mathbf{ds}} \times \vec{\mathbf{r}}}{r^3}$$

If we look at this expression, although for a general point in space it involves a very complicated integral over the circuit, it is clear that $\vec{\mathbf{B}}$ is everywhere proportional to i. That is, if we double the current, $\vec{\mathbf{B}}$ at every point in space will also double, all other things being the same. It then follows that the magnetic flux ϕ itself must also be proportional to i, since it is the surface integral of $\vec{\mathbf{B}}$, and $\vec{\mathbf{B}}$ is everywhere proportional to i.

That is, we must have $\phi = Li$, where L is a ***constant*** for a given arrangement of the wires of the circuit. If we change the geometry of the circuit (i.e., suppose we halve the radius of the circle in our sketch), we will change L, but for a ***given*** geometry, L does not change. The quantity L is called the self-inductance of the circuit, or simply the inductance. From its definition, we can show that the inductance has dimensions of μ_0 times length. We give an estimate of L for a single loop of wire below.

But first, let us write down the equation that governs the time evolution of i. If $\phi = Li$, then the time rate of change of ϕ is just $L\, di/dt$, so that we have from Faraday's Law

$$\oint \vec{\mathbf{E}} \cdot \vec{\mathbf{ds}} = -\varepsilon + iR = -\frac{d\phi}{dt} = -L\frac{di}{dt} \tag{1}$$

If we divide equation (1) by L, and rearrange terms, we find that the equation that determines the behavior of i is $di/dt + (R/L)i = \varepsilon/L$. The solution to this equation given our initial conditions is $i(t) = (\varepsilon/R)(1 - e^{-Rt/L})$ [see Giancoli, equation 30-9, p. 762]. This solution for $i(t)$ reduces to what we expect as t gets very large, ε/R, but also shows a continuous rise of the current from zero at $t = 0$ to this final value, in a characteristic time $\tau_L = L/R$ (τ_L is called the *inductive time constant*). This is the effect of having a non-zero inductance in a circuit, i.e., of taking into account the induced electric fields due to time changing $\vec{\mathbf{B}}$ fields. And this is what we expect from Lenz's Law — the reaction of the system is to try to keep things the same, that is to delay the build-up of current (or its decay, if we already have a current flowing in the circuit).

Kirchhoff's Second "Law" Modified for Inductors

We can write the governing equation for $i(t)$ from above (equation (1)) as

$$+\varepsilon - iR - L\frac{di}{dt} = \Delta V_i = 0 \tag{2}$$

where we have now cast it in a form that "looks like" a version of Kirchhoff's Second Law, namely that the sum of the potential drops around a circuit is zero (we are still moving counter-clockwise around the circuit, but the overall sign changes from equation (1) to (2) because we are now adding up changes in electric "potential").

Our text (Giancoli) chooses to approach circuits with inductance by preserving "Kirchhoff's Second Law," or the loop theorem, by specifying the "potential drop" across an inductor. To get the correct equation, Giancoli must make an additional "rule" for inductors as follows:

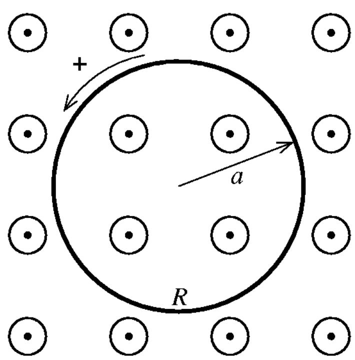
Inductors: If an inductor is traversed in the direction moving with the current, the change in potential is $-L\,di/dt$; if it is traversed in the direction opposite the current, the change in potential is $+L\,di/dt$.

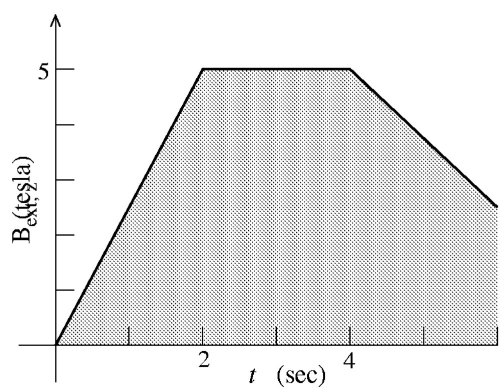
Although Giancoli never explicitly states this rule, it is implicit in his use of the "Loop Theorem" in sections 30-4, 30-5, and 30-6.

Use of this formalism will give the correct equations. However, the continued use of Kirchhoff's Second Law with this additional rule is **MISLEADING** at best, and at some level **DEAD WRONG** in terms of the physics, for the following reasons. Kirchhoff's Second Law was originally based on the fact that the integral of $\vec{\mathbf{E}}$ around a closed loop was zero. With time-changing magnetic fields, this is no longer so, and thus the sum of the "potential drops" around the circuit, if we take that to mean the negative of the closed loop integral of $\vec{\mathbf{E}}$, is ***no longer zero*** — in fact it is $+L\,di/dt$. As do many introductory texts, Giancoli brings the $L\,di/dt$ term to the other side of the equation, adds it to the negative of the closed loop of $\vec{\mathbf{E}}$, and ascribes it to a "potential drop" across the inductor.

This approach gives the right equations, but it sure confuses the physics. In particular, having a "potential drop" across the inductor of $-L\,di/dt$ implies that there is an electric field in the inductor such that the integral of $\vec{\mathbf{E}}$ through the inductor is equal in magnitude to $L\,di/dt$. **This is not always, or even usually, true**, as in our example above (the integral of $\vec{\mathbf{E}}$ through our "one-loop" inductor above is **zero**, **NOT** $L\,di/dt$).

The fact that $\vec{\mathbf{E}}$ is zero in our "one-loop inductor" above may confuse you, and for good reason. You have developed some intuition about induced electric fields, based on the kinds of Faraday's Law problems we have been doing up to now. The fact is, quite often in the past when we have had time-changing magnetic fields, we have had an electric field right where $d\vec{\mathbf{B}}/dt$ was non-zero. That fact would make you think that Giancoli is right, that there is an electric field right there in the inductor, and thus a potential drop across it. What has changed in our circuit above to make $\vec{\mathbf{E}}$ zero in our "one-loop inductor," even though there is a time-changing magnetic field through it? This is a very subtle point, and the source of endless confusion, so let's look at it carefully.



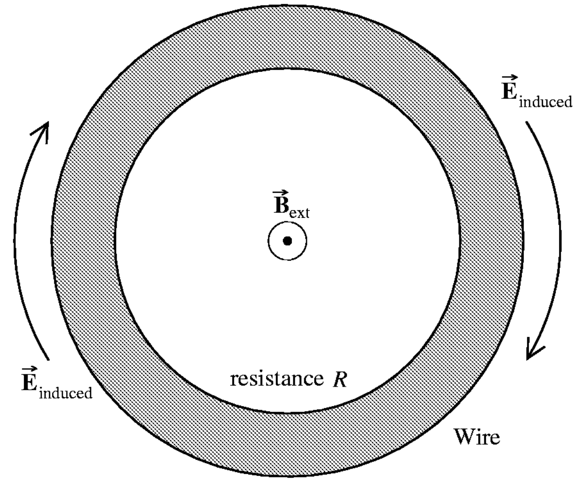


Our intuition that there should be an electric field in an inductor is based on doing problems like that shown in the sketch to the right. We have a loop of wire of radius a and total resistance R, immersed in an external magnetic field which is out of the page and increasing with time as shown. In considering this circuit, unlike our "one-loop" inductor above, we *neglect* the magnetic field due to the currents in the wire itself, assuming that $\vec{\mathbf{B}}_{\text{ext}}$ is much greater than that field, and consider *only* the effects of the external field. The conclusions we arrive at here can be applied as well to the self-inductance case.

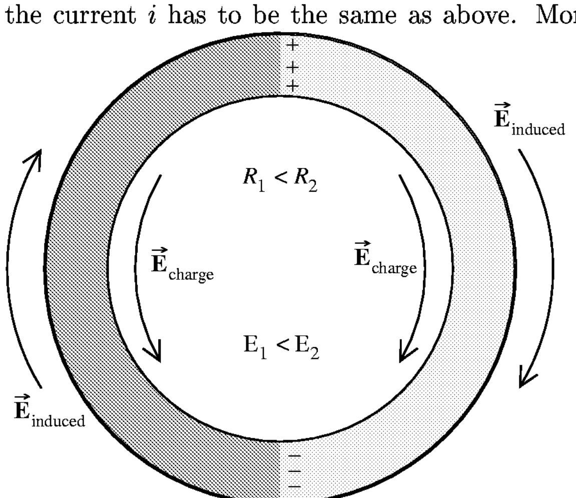
The changing external magnetic field will give rise to an induced electric field in the loop of wire, with a line integral which is equal to $-\pi a^2(d\vec{\mathbf{B}}_{\text{ext}}/dt)$. This induced electric field is azimuthal and uniformly distributed around the loop (see sketch). We have from Faraday's Law that

$$\begin{aligned} 2\pi a\vec{\mathbf{E}}_{\text{induced}} &= -\pi a^2\frac{d\vec{\mathbf{B}}_{\text{ext}}}{dt} \\ \Rightarrow \vec{\mathbf{E}}_{\text{induced}} &= -\frac{a}{2}\frac{d\vec{\mathbf{B}}_{\text{ext}}}{dt} \end{aligned}$$

Thus if the resistance is distributed uniformly around our wire loop, we get a uniform $\vec{\mathbf{E}}_{\text{induced}}$ in the loop which is the same at every point in the wire loop, and circulating clockwise for $\vec{\mathbf{B}}_{\text{ext}}$ increasing in time. This electric field causes a current, with the current density given by $\vec{\mathbf{j}} = \vec{\mathbf{E}}_{\text{induced}}/r$ (the microscopic form of Ohm's Law). The total current in the wire loop will be the total "potential drop" around the loop divided by the resistance R (the macroscopic form of Ohm's Law), or $2\pi a \vec{\mathbf{E}}_{\text{induced}}/R$. This current will circulate clockwise in the same sense as $\vec{\mathbf{E}}_{\text{induced}}$. Thus if the resistance is distributed uniformly around our wire loop, we get a uniform $\vec{\mathbf{E}}_{\text{induced}}$ in the loop which is the same at every point in the wire loop, and circulating clockwise for increasing $\vec{\mathbf{B}}_{\text{ext}}$.



But what happens to the electric field if we *don't* distribute the resistance uniformly around the loop. For example, let's make the left half of our loop out of wire with resistance R_1, and the right half the loop out of wire with resistance R_2, with $R = R_1 + R_2$, so that we have the same total resistance as before (see sketch next page). Let's furthermore assume that $R_1 < R_2$. NOTICE some similarity with the demo I (Walter Lewin) did in lecture on March 15, 2002 (read my Lecture Supplement). How is the electric field distributed around the loop of wire now? First of all, the *emf* in the circuit is the same as above, as is the total resistance, so that the current i has to be the same as above. Moreover it must be the same on both sides of the loop, by charge conservation. But the electric field in the left half of the wire loop ($\vec{\mathbf{E}}_1$) must now be ***different*** from that in the right half ($\vec{\mathbf{E}}_2$).



This is so because the line integral of the electric field on the left side, over the left side, is $\pi a \mathrm{E}_1$, and this must be (from Ohm's Law) equal to iR_1. Similarly, $\pi a \mathrm{E}_2 = iR_2$. Thus $\mathrm{E}_1/\mathrm{E}_2 = R_1/R_2$, and therefore $\mathrm{E}_1 < \mathrm{E}_2$, since $R_1 < R_2$. And this makes sense. We must get the same current on both sides, even though the resistances are different. We do this by adjusting the electric field on the side with the smaller resistance to ***be*** smaller. Because the resistance is also smaller, we produce the same current as on the opposing side with this smaller electric field.

But what happened to our uniform electric field? Well, there are ***two*** ways to produce electric fields — one from time-changing magnetic fields, the other from electric charges. Nature accomplishes the reduction in $\vec{\mathbf{E}}_1$ compared to $\vec{\mathbf{E}}_2$ by charging up the junctions separating the wire segments (see sketch above), positive on top and negative on bottom. The total electric field is the sum of the electric field induced by the changing external magnetic field ($\vec{\mathbf{E}}_{\text{induced}}$, as indicated in the sketch above, still clockwise), and the electric field associated with the charging at the junctions ($\vec{\mathbf{E}}_{\text{charge}}$, as indicated in the sketch, going from positive charge to negative charge, as is always true for fields produced by charges). It is clear that the addition of these two contributions to the electric field will reduce the total electric field on the left and enhance it on the right. The field $\vec{\mathbf{E}}_1$ will always be clockwise (as it must be to produce clockwise current flow), but it can be made arbitrarily small by making $R_1 \ll R_2$. However, we still always have the integral of $\vec{\mathbf{E}}$ over the complete closed loop equal to $-\pi a^2 (d\vec{\mathbf{B}}_{\text{ext}}/dt)$, as Faraday's Law demands.

Thus we see that we can make a non-uniform electric field in an inductor by using non-uniform resistances, even though our intuition tells us (correctly) that the *induced* electric field should be uniform at a given radius. The reality is that there is *another* way to produce electric fields, namely from charges, and Nature uses that fact as needed. All that Faraday's Law tells us is that the line integral of $\vec{\mathbf{E}}$ around a closed loop is equal to the negative time rate of change of the magnetic flux through the enclosed surface. It doesn't tell us at what locations the $\vec{\mathbf{E}}$ field is non-zero around the loop, and it may be non-zero (or zero!) in unexpected places. The field in wire making up the "one-loop inductor" above was zero (or at least very small), with the significant fields occurring only in the resistor and the battery, for exactly the sort of reason we have considered here.

One final point. Suppose you put the probes of a voltmeter across the terminals of an inductor (with very small resistance) in a circuit. What will you measure? What you will measure on the meter of the voltmeter is a "voltage drop" of $L\,di/dt$. But that is not because there is an electric field in the inductor! It is because putting the voltmeter in the circuit will result in a time changing magnetic flux through the voltmeter circuit, consisting of the inductor, the voltmeter leads, and the large internal resistor in the voltmeter (see my Lecture Supplement of March 15, 2002). A current will flow in the voltmeter circuit because there will be an electric field in the large internal resistance of the voltmeter, with a potential drop across that resistor of $L\,di/dt$, by Faraday's Law applied to the voltmeter circuit, and that is what the voltmeter will read. The voltmeter as usual gives you a measure of the potential drop across its own internal resistance, but this is *not* a measure of the potential drop across the inductor. It is a measure of the time rate of change of magnetic flux in the voltmeter circuit! As before, there is only a very small electric field in the inductor if it has a very small resistance compared to other resistances in the circuit.

If you find all this confusing, you are in good company. This is one of the most difficult and subtle topics in this course — it trips up experts all the time. Not easy!