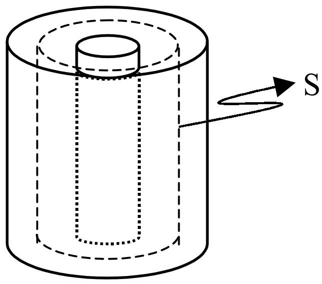
Primer parcial de electromagnetismo - curso 2006

- Solución -

Problema 1

a) Ley de Gauss en S: $D(r)2\pi rL = Q$



$$\Rightarrow \vec{D}(r) = \frac{Q}{2\pi L}\frac{1}{r}\hat{e}_r, \quad \vec{E}(r) = \frac{\vec{D}(r)}{\varepsilon} = \frac{Q}{2\pi\varepsilon L}\frac{1}{r}\hat{e}_r$$

$$V(2a) - V(a) = -\int_a^{2a} E(r)dr \;\rightarrow\; -V = -\frac{Q}{2\pi\varepsilon L}\mathrm{Ln}(r)\Big|_a^{2a} = -\frac{Q}{2\pi\varepsilon L}\mathrm{Ln}(2)$$

$$\therefore\; C = \frac{Q}{V} = \frac{2\pi\varepsilon}{\mathrm{Ln}(2)}L$$

b)

$$\left.\begin{aligned} u &= \frac{1}{2}\vec{E}.\vec{D} = \frac{1}{2}\varepsilon E^2 \\ E &= \frac{Q}{2\pi\varepsilon L}\frac{1}{r} = \frac{V}{\mathrm{Ln}(2)}\frac{1}{r} \end{aligned}\right\} \Rightarrow u = \frac{1}{2}\varepsilon\left(\frac{V}{\mathrm{Ln}(2)}\right)^2\frac{1}{r^2}$$

c) En equilibrio: $F_E = mg$

$$m = \eta\pi\left[(2a)^2 - a^2\right]h = 3\pi a^2\eta h$$

$$\vec{F}_E = \left.\frac{\partial U}{\partial h}\right|_V \hat{k} \qquad \text{con: } U = U_{vacío} + U_{diel.}$$

$$U_{vacío} = \int_a^{2a} u_v 2\pi r dr(L-h) = \int_a^{2a}\frac{1}{2}\varepsilon_0 E^2 2\pi r(L-h)dr = \frac{\varepsilon_0\pi(L-h)V^2}{\left[\mathrm{Ln}(2)\right]^2}\int_a^{2a}\frac{dr}{r}$$

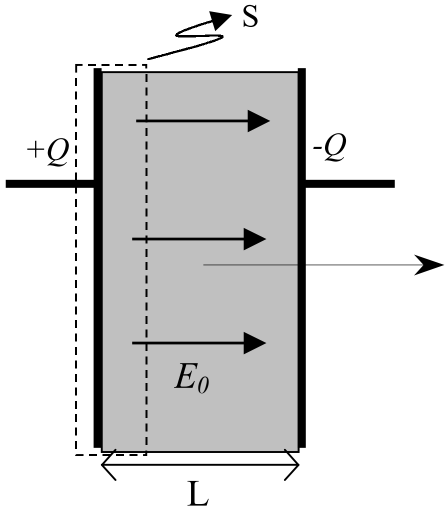
$$\Rightarrow U_{vacío} = \frac{\varepsilon_0\pi(L-h)V^2}{\mathrm{Ln}(2)}.$$

$$U_{diel.} = \int_a^{2a} u_d 2\pi r dr h = \frac{\varepsilon\pi h V^2}{\mathrm{Ln}(2)}.$$

$$\Rightarrow U = \frac{\pi V^2}{\mathrm{Ln}(2)}\left[(\varepsilon-\varepsilon_0)h + \varepsilon_0 L\right] \quad \therefore\; \vec{F}_E = \frac{\pi V^2}{\mathrm{Ln}(2)}(\varepsilon-\varepsilon_0)\hat{k}$$

Equilibrio: $\dfrac{\pi V^2}{\mathrm{Ln}(2)}(\varepsilon-\varepsilon_0) = 3\pi a^2\eta h g \;\Rightarrow\; h = \dfrac{(\varepsilon-\varepsilon_0)V^2}{3\mathrm{Ln}(2)a^2\eta g}$

Problema 2



a) $L << R \Rightarrow \begin{cases} \vec{E}_0 = E_0\hat{k} \\ E = 0 \text{ afuera.} \end{cases}$

Si se aplica la ley de Gauss en la superficie *S* de forma cilíndrica y que rodea a la placa con carga +Q:

$D_0 \pi R^2 = Q \quad \Rightarrow \quad \varepsilon E_0 \pi R^2 = Q \quad \Rightarrow$

$$\begin{cases} \vec{E}_0 = \dfrac{Q}{\pi\varepsilon R^2}\hat{k} \\ \vec{P} = \chi \vec{E}_0 = (\varepsilon - \varepsilon_0)\dfrac{Q}{\pi\varepsilon R^2}\hat{k} \end{cases}$$

Sobre la placa con +Q: $\sigma_p^+ = \vec{P}.(-\hat{k}) = -\dfrac{(\varepsilon-\varepsilon_0)Q}{\pi\varepsilon R^2}\hat{k}$.

Sobre la placa con -Q: $\sigma_p^- = \vec{P}.\hat{k} = \dfrac{(\varepsilon-\varepsilon_0)Q}{\pi\varepsilon R^2}\hat{k}$.

b) Sean (r,θ,ϕ) coordenadas esféricas con origen en el centro de la cavidad, como hay simetría en ϕ el potencial eléctrico dependerá sólo de r y θ: $\Phi = \Phi(r,\theta)$.

Si: $\begin{cases} \Phi_1 : \text{ potencial en la cavidad.} \\ \Phi_2 : \text{ potencial en el dieléctrico.} \end{cases}$ Se cumple: $\nabla^2\Phi_1 = \nabla^2\Phi_2 = 0$.

Sean: $\begin{cases} \Phi_1(r,\theta) = A_0 + B_0 r^{-1} + (A_1 r + B_1 r^{-2})\cos\theta \\ \Phi_2(r,\theta) = A_0' + B_0' r^{-1} + (A_1' r + B_1' r^{-2})\cos\theta \end{cases}$

* $B_0 = B_0' = 0$, porque no hay carga neta.
* $B_1 = 0$, porque Φ_1 es continuo en la cavidad.

$$\Rightarrow \Phi_1(r,\theta) = A_0 + A_1 r\cos\theta$$
$$\Phi_2(r,\theta) = A_0' + (A_1' r + B_1' r^{-2})\cos\theta$$

Condiciones de borde:

* Si $r >> a$ debe ser: $\vec{E} \cong E_0\hat{k} \Rightarrow \Phi_2 \cong -E_0 z + cte = -E_0 r\cos\theta + cte$

$\Rightarrow A_1' = -E_0 = -\dfrac{Q}{\pi\varepsilon R^2}$.

* $\Phi_1(r=a,\theta) = \Phi_2(r=a,\theta) \quad \forall\theta \Rightarrow A_0 + A_1 a\cos\theta = A_0' + (A_1' a + B_1' a^{-2})\cos\theta \quad \forall\theta$

$$\Rightarrow \begin{cases} A_0 = A_0' & \rightarrow \quad \text{se puede elegir: } A_0 = A_0' = 0. \\ A_1 a = A_1' a + B_1' a^{-2} & \rightarrow \quad A_1 = A_1' + B_1' a^{-3} \quad (1) \end{cases}$$

* $\vec{D}_{2n}\Big|_a - \vec{D}_{1n}\Big|_a = \sigma_L = 0$

Es: $\vec{E}_1 = -\dfrac{\partial \Phi_1}{\partial r}\hat{e}_r - \dfrac{1}{r}\dfrac{\partial \Phi_1}{\partial \theta}\hat{e}_\theta = -A_1\cos\theta\,\hat{e}_r + A_1 \operatorname{sen}\theta\,\hat{e}_\theta = -A_1\,\hat{k}$

$\vec{E}_2 = -(A_1' - 2B_1' r^{-3})\cos\theta\,\hat{e}_r + (A_1' + B_1' r^{-3})\operatorname{sen}\theta\,\hat{e}_\theta$

$\vec{D}_1 = \varepsilon_0 \vec{E}_1 \qquad \vec{D}_2 = \varepsilon \vec{E}_2 \qquad \hat{n} \equiv \hat{e}_r$

$\Rightarrow -\varepsilon(A_1' - 2B_1' a^{-3})\cos\theta = -\varepsilon_0 A_1 \cos\theta \;\rightarrow\; \varepsilon_0 A_1 = \varepsilon A_1' - 2\varepsilon B_1' a^{-3} \qquad (2)$

De (1) y (2): $\varepsilon_0 A_1' + \varepsilon_0 B_1' a^{-3} = \varepsilon A_1' - 2\varepsilon B_1' a^{-3} \;\Rightarrow\; B_1' = \dfrac{\varepsilon - \varepsilon_0}{2\varepsilon + \varepsilon_0} a^3 A_1'$

$$B_1' = -\frac{\varepsilon - \varepsilon_0}{2\varepsilon + \varepsilon_0} a^3 E_0$$

De (1): $A_1 = -E_0 - \dfrac{\varepsilon - \varepsilon_0}{2\varepsilon + \varepsilon_0} E_0 = -\dfrac{3\varepsilon}{2\varepsilon + \varepsilon_0} E_0$

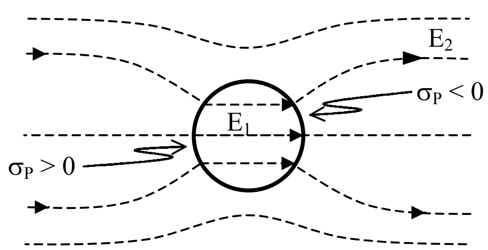
$\therefore\; \vec{E}_1 = \dfrac{3\varepsilon}{2\varepsilon + \varepsilon_0} E_0 \hat{k}$

c) $\sigma_p = \vec{P}_2\Big|_a . \hat{n} = \vec{P}_2\Big|_a .(-\hat{e}_r)$

$\vec{P}_2\Big|_a = \chi \vec{E}_2\Big|_a = (\varepsilon - \varepsilon_0)\,\vec{E}_2\Big|_a$

$\Rightarrow \sigma_P = -(\varepsilon - \varepsilon_0)(A_1' - 2B_1' a^{-3})\cos\theta\,\hat{e}_r.(-\hat{e}_r) = (\varepsilon - \varepsilon_0)(A_1' - 2B_1' a^{-3})\cos\theta$

$$= (\varepsilon - \varepsilon_0)E_0\left(-1 + \frac{2(\varepsilon - \varepsilon_0)}{2\varepsilon + \varepsilon_0}\right)\cos\theta$$



$\therefore\; \sigma_P = -3\dfrac{\varepsilon_0(\varepsilon - \varepsilon_0)}{2\varepsilon + \varepsilon_0} E_0 \cos\theta$